Computer Algebra in Service Courses

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Nowadays there is much interest in how to use computer algebra in the teaching of mathematics. Much attention is given to how maths courses can be made more interesting, more stimulating for the students. For me, it is rather astonishing to see that hardly any attention is paid to the main use of computer algebra as a tool for doing mathematics. As such, it has a big impact on what topics should be taught in math courses. Concentrating on skills for doing mathematical computations is no longer necessary, that can be left to the computer. Real mathematical understanding becomes more important. My experiences can be summarised in the following two observations:

1. Computer algebra can be used to trivialise the standard service courses in mathematics.
2. Good use of computer algebra requires a better understanding of mathematics.

These two conflicting statements will be illustrated by some examples using *Mathematica*. Then some remarks will be made about the future of the service courses in mathematics due to the availability of computer algebra and other mathematical software.

1. Some simple examples

In a traditional course, computing the following integral by hand is rather complicated:

$$\int_0^\infty \frac{dx}{\left(x^3+1\right)^2}$$

However, with computer algebra the problem reduces to a simple command:

```
In[1]:= Integrate [1/(x^3+1)^2,
        {x,0,Infinity}]
```

Out[1] = $\dfrac{4\pi}{9\sqrt{3}}$

The same applies to the differential equation

$x'' - 3x' + 2x = \sin t,\ x(0) = 3,\ x'(0)=1$:

```
In[2]:= DSolve
        [{x''[t] - 3x'[t] + 2x[t] == Sin[t],
        x[0]==3, x'[0]==1}, x[t], t]
```

Out[2]= $\{\{x(t) \to \dfrac{45e^t - 18e^{2t} + 3\cos(t) + \sin(t)}{10}\}\}$

These two examples can be computed by hand. However, for quite a lot of similar problems analytical solutions do not exist. With computer algebra, that is not a problem. We can find numerical solutions just as easy as analytical ones, as the following example shows. First we solve a differential equation numerically, then assign the resulting numerical function to the name *f* and plot the graph of *f* as shown in figure 1.

```
In[3]:= NDSolve
        [{x''[t]+(t^2-1)x'[t]+tx[t]==Sin[t],
        x[0]==3, x'[0]==1}, x, {t,0,5}]
```

Out[3]= {{*x* → *InterpolatingFunction*[{{0., 5.}}, <>]}}

```
In[4]:= f = x /.%[[1]]; Plot[f[t],{t,0,5}]
```

Out[4]= < Graphics>

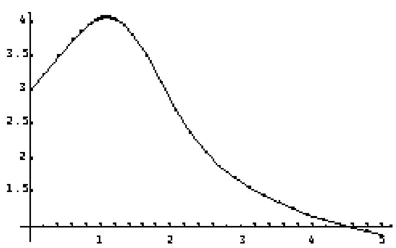

Figure 1: Numerical solution of a differential equation

This plot suggests that there is a maximum value somewhere near 1. This maximum can easily be found.

```
In[5]:= {x,f[x]}/.FindRoot[f'[x]==0,{x,1}]
```

Out[5]= {1.09905, 4.05818}

These examples show that technically complicated results can be obtained just by using some straightforward commands of the computer algebra package and that not much mathematical understanding is required.

However, not all traditional exercises reduce to only one command of the package. As an example, let us consider the topic of line integrals. Suppose we are given a function $f(x,y)=x^2+3yz$ and a curve $(x,y,z)=(t,t^2,t^3)$ where t runs from 0 to 1, and that we have to integrate the function f over the curve. There is no standard command for doing this, but each of the steps can easily be performed using the package. We follow the recipe from the traditional calculus courses. We substitute the parametrization into the expression, compute the speed with which we move along the curve, multiply and integrate from 0 to 1.

```
In[6]:=  f = x^2 + 3 y z; par = {x -> t,
      y -> t^2, z -> t^3};
```

In[7]:= D[Last /@ par, t]

Out[7]= $\{1, 2t, 3t^2\}$

```
In[8]:= Sqrt[ Plus @@ (%2) ]
```

Out[8]= $\sqrt{1+4t^2+9t^4}$

```
In[9]:= (f /.  par} %
```

Out[9]= $\sqrt{1+4t^2+9t^4}\,(t^2+3t^5)$

```
In[10]:= NIntegrate[%, {t,0,1}]
```

Out[10]= 2.37102

In order to be able to do these calculations by computer, one needs some skill in the typical commands of the package. But a teacher or a clever user can combine all these steps into one new command, thereby reducing computing lineintegrals to pressing buttons:

```
In[11]:= NLineIntegrate
      [f_,par_,{t_,a_,b_}, opts___] :=
      NIntegrate[Evaluate[(exp /. par)
      Sqrt[ Plus@@ (D[Last /@ par, t]2)]],
      {t,a,b}, opts]
In[12]:= NLineIntegrate [f, {t, 0, 1}]
```

Out[12]= 2.37102

```
In[13]:=NLineIntegrate
      [Sin[xy] + Exp [-x^2 u^2 v^2 w^2],
      {x -> t, y -> 2t, u -> 3 + Sin[t],
      v -> Cos[t], w -> t2}, {t, -1, 2}]
```

Out[13]= 6.95477

2. A more complicated example

Doing trigonometric computations with pen and paper often requires a clever application of many trigonometric formula that must be known by heart. Computer algebra is powerful, but not clever and therefore even with computer algebra at hand we must be clever ourselves. However, the tools available in computer algebra differ from the formula that old-fashioned mathematicians know by heart. In *Mathematica*, they have been replaced by three functions:

- TrigReduce, for reducing the degree of the polynomial, considered as a polynomial in sine and cosine functions, to 1,
- TrigFactor, for writing the trigonometric polynomial as a product of (in most cases) irreducible trigonometric polynomials,
- TrigExpand, to reduce the arguments of the trigonometric functions in the expression to a single variable.

Now we are going to use these commands in proving the following curious formula:

$$\sin(50^\circ)\sin(60^\circ)\sin(70^\circ) = \sin(40^\circ)\sin^2(80^\circ).$$

All arguments of the trigonometric functions are multiples of 10º. So we consider the expression and try to show that it equals 0 when substituting 10º for x.

$$\sin(5x)\sin(6x)\sin(7x) - \sin(4x)\sin^2(8x)$$

Entering this expression in *Mathematica*, substituting $\pi/18$ for x and using the Simplify command, unfortunately does not yield the output 0. So *Mathematica* is unable to demonstrate this relation by itself. Of course, we can compute the numerical value as accurately as we want, which gives some evidence that the relation indeed holds.

One way we can solve the problem is reducing the expression to a polynomial in $\sin(x)$ and $\cos(x)$ and then try to simplify this polynomial using some relations between $\sin(x)$ and $\cos(x)$ that hold for $x=\pi/18$.

```
In[4] := exp = Sin[5x] Sin[6x] Sin[7x] -
      Sin[4x] Sin[8x]^2
```

Out[4] = $\sin(5x)\sin(6x)\sin(7x) - \sin(4x)\sin^2(8x)$

```
In[5] := TrigExpand [ exp ]:
```

This gives rise to a fairly complicated polynomial. But it can be reduced, using the properties that $\sin^2(x)+\cos^2(x)=1$ and $\sin(30°)=\sin(3x)=½$. These relations are given in the following command as the argument of the *Mathematica* AlgebraicRules function.

```
In[6]:= %/. AlgebraicRules
      [{Sin[x]^2 + Cos[x]^2 == 1,
      TrigExpand[Sin[3 x]] == 1/2}]
Out[6]= 0
```

So now the formula has been shown. Most mathematicians do not like this proof and prefer to do some pen and paper calculations. Anyway, the above approach in showing the formula is very unsuitable for pen and paper, but straightforward on a computer.

Some points for discussion

A. *The content of traditional service courses in mathematics has been determined by the needs of users of mathematics in the pre-computer age.*

From the first examples we see that most of the traditional exercises and even more complicated ones can be done much easier by computer than by pen and paper. Will an engineer ever use pen and paper techniques once he / she realises that a computer can do it faster and better?

Many mathematicians with a rich experience in applying pen and paper techniques complain that doing the a computation with a computer costs them more time than doing the same computation by hand. We also see that one needs skill in working with a computer algebra package in an adequate manner, just as one needs skill in applying pen and paper techniques. One needs to invest time in getting used to a package, and only then will it be profitable.

Thirdly we notice that the approach for solving a problem with computer algebra or by hand often is different. A good example of this was the trigonometry. Instead of clever use of trigonometric formula we used a more general straightforward technique of factorisation of trigonometric polynomials, or by reducing polynomials modulo polynomial relations, techniques that would never have been used by hand.

B. A service course in mathematics in which the computational work is done by computer algebra will have a different content and emphasis on different skills than a traditional service course.

One of the objections that is often heard is that pressing buttons for solving mathematical problems has nothing to do with mathematics. But in my university most of the service courses (unofficially) stress reproducing mathematical recipes for solving standard exercises that can be applied by the students without any understanding. The average student is only interested in how to solve the exercise, not in the mathematical theory behind it. What is then the difference with using the computer for solving these exercises? Students like using the computer and make less mistakes!

Consider equations with no exact analytical solution. In *Mathematica* these can be solved numerically using the NDSolve command. So no understanding and no detailed knowledge of the package is required.

C. The use of computer algebra increases the level of what can be done without much understanding to that of the second or in the near future third or even fourth year university.

The availability of computer algebra strongly changes the mathematical needs of scientists and engineers and even the way we do and apply mathematics. Therefore the contents of service courses in mathematics have to be reconsidered. How and in what direction?

Roughly speaking, I think we have to consider two cases: disciplines in which mathematics plays a prominent role (only a few) and disciplines in which mathematics is used on a relatively low level (most). In the latter category the knowledge of mathematics and the use that is made of mathematics is not very widespread. What is needed is available in the present computer algebra packages.

D. Service courses in mathematics that concentrate on how to solve exercises can be transformed into courses on how to use a computer algebra package.

The advantage of this is that the success rate of the courses will increase tremendously. To the standards of educators and politicians this means that the quality of the mathematics courses will suddenly be much higher. To the standards of mathematicians these courses are very poor. Mathematics is about concepts and reasoning, not about pressing buttons.

Another consequence is that once these faculties realise that the standard mathematical software packages completely satisfy their needs, they might be inclined to abolish the mathematics courses completely. At some places this process has started already. It might very well be that in a few years time for most faculties there will be no need for service teaching of mathematics. The way most people will make use of mathematics is by computer algebra and looking at the computer output, not by doing computations themselves.

On the other hand, the role of mathematics in many areas is increasing. We need people who mathematically understand what they are doing and know the limitations of what can be done with a computer. Those students need mathematics courses with emphasis on insight and reasoning.

E. For faculties that need a real understanding of mathematics completely new courses have to be developed, with emphasis on concepts and reasoning and integrated use of computer algebra as a tool for doing the computations.

Developing such courses is a challenge. We have quite a lot of experience in courses with recipes on how to solve a problem, but not with courses that stress understanding. Understanding is difficult to examine. Too often students think they understand mathematics when they only know how to solve the exercises.

F. We have to accept that mathematics is difficult and that not everyone has the capacities for being successful. But gifted students need better mathematics teaching than is presented nowadays. Computer algebra packages are a very valuable tool for doing so.

This observation is in strong contrast to the developments in the area of mathematical education. There seems to be something like a ‘mathematics for all' movement, where mathematics must be fun. I agree that mathematics is fun, but the fun of finding an elegant solution, an efficient algorithm etc, can only be appreciated when it is based on a sound foundation of mathematical understanding.

Editor's note: This is a summary of a thought-provoking paper presented at the second IMA conference on Mathematical Education of Engineers, held at Loughborough University from 7-9 April. Thanks to Fred for providing a transcript, and to Grant Keady for help with its preparation.