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On the order of the recurrence produced by the method of creative telescoping

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Abstract

We present an algorithm which computes a non-trivial lower bound for the order of the minimal telescoper for a given hypergeometric term. The combination of this algorithm and techniques from indefinite summation leads to an efficiency improvement in Zeilberger's algorithm. We also describe a Maple implementation, and conduct experiments which show the improvement that it makes in the construction of the telescopers.

Keywords: Zeilberger's algorithm; Hypergeometric term; Minimal telescoper; Lower bound; Additive decomposition

1. Preliminaries

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0, the variables n, k be integer-valued, and E_n, E_k be the corresponding shift operators, acting on functions of n and k, by $E_n f(n,k) = f(n+1,k)$, $E_k f(n,k) = f(n,k+1)$. A $\mathbb{K}$-valued function $t(k)$ is a

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hypergeometric term of k over $\mathbb{K}$ if the consecutive term ratio $\mathscr{C}_k(t) = E_k t/t$ is a rational function of k over $\mathbb{K}$. The rational function $\mathscr{C}_k(t)$ is called the *certificate* of $t(k)$. A $\mathbb{K}$-valued function $T(n,k)$ is a hypergeometric term of two variables n and k if the two consecutive term ratios $\mathscr{C}_n(T) = E_n T/T$, and $\mathscr{C}_k(T) = E_k T/T$ are rational functions of n and k over $\mathbb{K}$. The rational functions $\mathscr{C}_n(T)$, $\mathscr{C}_k(T)$ are called the n-certificate and the k-certificate of T, respectively. Given a hypergeometric term $T(n,k)$ as input, Zeilberger's algorithm [14,16,17] (which we denote hereafter as $\mathscr{Z}$) constructs for $T(n,k)$ a *Z-pair* (L, G), provided that such a pair exists. The computed Z-pair consists of L, a linear recurrence operator of order ρ with coefficients which are polynomials of n over $\mathbb{K}$, i.e.,

$$L = a_\rho(n)E_n^\rho + \cdots + a_1(n)E_n^1 + a_0(n)E_n^0, \quad a_i(n) \in \mathbb{K}[n] \tag{1}$$

and a hypergeometric term $G(n,k)$ such that

$$LT(n,k) = (E_k - 1)G(n,k). \tag{2}$$

The k-free operator L is called a *telescoper*. It is noteworthy that the problem of establishing a necessary and sufficient condition for the applicability of $\mathscr{Z}$ to $T(n,k)$ is solved and presented in [1,2] (the well-known *fundamental theorem* [16,17] only provides a sufficient condition). It is proven in [17] that if there exists a Z-pair for $T(n,k)$, then $\mathscr{Z}$ terminates with one of the Z-pairs, and the telescoper L in the returned Z-pair is of minimal order. The computed telescoper L is unique up to a left-hand factor $P(n) \in \mathbb{K}[n]$, and we name it *the minimal telescoper* [17].

$\mathscr{Z}$ has a wide range of applications which include finding closed forms of definite sums of hypergeometric terms, verification of combinatorial identities, and asymptotic estimation [14,17,13].

The algorithm uses an *item-by-item examination* on the order ρ of the operator L of the form (1). It starts with the value of 0 for ρ and increases ρ until it is successful in finding a Z-pair (L, G) for T. In other words, a lower bound for ρ is 0. As a consequence, we waste resources trying to compute without success a telescoper of ord $L < \rho$, where ρ is the order of the minimal telescoper.

In this paper, we present an algorithm which computes an improved non-zero lower bound for the order of the telescopers. The general approach of the algorithm can be described as follows: for a given hypergeometric term $T(n,k)$, apply the algorithm which solves the additive decomposition problem to T w.r.t. k to obtain a pair of similar hypergeometric terms $T_1(n,k), T_2(n,k)$ such that $T = (E_k - 1)T_1 + T_2$, and either $T_2 = 0$ (i.e., T is k-summable) or T_2 has some specific features each of which ensures that T_2 is not k-summable. In the former case, it is evident that $\mathscr{Z}$ is applicable to T and the minimal telescoper for T is 1. In the latter case, it is easy to show that a telescoper for T exists if and only if a telescoper for T_2 exists, and the sets of telescopers for T and T_2 are the same. We consider recurrence operators $M \in \mathbb{K}[n, E_n]$, called *crushing operators*, with the property that if M is a crushing operator for T_2, then MT_2 does not have at least one of the specific features that T_2 does (this does not guarantee that MT_2 is k-summable, though). It follows that the order of the minimal telescoper for T_2 is always greater than or equal to that of a minimal crushing operator M for T_2. We then describe an algorithm which computes a lower bound $\mu > 0$ for the order of the crushing operators for T_2. This value is automatically also a lower bound for the order of the telescopers for T.

When $T(n,k)$ is not k-summable and the algorithm is used in combination with the algorithm which determines the applicability of $\mathscr{Z}$ to $T(n,k)$ [1,2], it allows one to use $\mathscr{Z}$ to compute a Z-pair only if the existence of such a pair is guaranteed, and in this case, one can use $\mu > 0$ as the starting value for the order of L, instead of 0. Let ρ be the order of the minimal telescoper L; since the computation of a lower bound μ is in general less expensive than that of telescopers of order $0, \ldots, \mu - 1$, especially when the computed value μ is close to ρ and ρ has a large value, this will lead to some efficiency improvement. Also, since T_2 is "simpler" than T in some sense and since the minimal telescopers for T and T_2 are the same, applying $\mathscr{Z}$ to T_2 instead of to T can provide some significant efficiency improvement (see Example 6).

Note that for the case where the hypergeometric term $T(n,k)$ is also a rational function, there is a direct algorithm which computes the minimal telescoper for T efficiently without using item-by-item examination [10].

The paper is organized in the following manner. In Section 2, we discuss some known results which are needed in subsequent sections. They include a description of the additive decomposition problem of hypergeometric terms [6,9], and a criterion for the applicability of $\mathscr{Z}$ [1,2]. The main result of Section 3 is a theorem which helps to compute a lower bound for the order of a minimal crushing operator. An algorithmic description for this theorem is presented in detail in Section 4. We conclude the paper with a description of an implementation of the algorithm in Section 5. Various examples are used to show the advantages of this implementation over an implementation of the original $\mathscr{Z}$.

Throughout the paper, $\mathbb{K}$ is an algebraically closed field of characteristic 0; $\mathbb{Z}$ and $\mathbb{N}$ denote the set of integers and non-negative integers, respectively. Following [14], we write $T_1(n,k) \sim T_2(n,k)$ if two non-zero hypergeometric terms $T_1(n,k)$ and $T_2(n,k)$ are *similar*, i.e., their ratio is a rational function of n and k.

A preliminary version of this paper has appeared as [4].

2. The additive decomposition problem and the existence of a telescoper

We begin this section with the notion of *Rational Normal Forms* (RNF) of a rational function [7]. This concept plays an important role in the follow-up algorithms.

Definition 1. Let $\mathbb{F}$ be a field of characteristic 0, and $R \in \mathbb{F}(k)$ be a non-zero rational function. If there are $f_1, f_2, v_1, v_2 \in \mathbb{F}[k]\backslash\{0\}$ such that

(i) $R = F \cdot \frac{E_k V}{V}$, where $F = \frac{f_1}{f_2}$, $V = \frac{v_1}{v_2}$, and $\gcd(v_1, v_2) = 1$,
(ii) $\gcd(f_1, E_k^h f_2) = 1$ for all $h \in \mathbb{Z}$,

then $F \cdot \frac{E_k V}{V}$ is an RNF of R.

The rational function F in (i) with property (ii) is called the *kernel* of the RNF. Note that every rational function has an RNF [9, Theorem 1] which in general is not unique.

2.1. The additive decomposition problem

For a hypergeometric term $T(k)$ of k over $\mathbb{F}$, the algorithm which solves the additive decomposition problem [6,9] constructs two hypergeometric terms $T_1(k),\ T_2(k)$ similar to $T(k)$ such that

(i)

$$T(k) = (E_k - 1)\, T_1(k) + T_2(k) \text{ and} \tag{3}$$

(ii) either $T_2 = 0$ or $\mathscr{C}_k(T_2)$ has an RNF

$$\frac{f_1}{f_2}\,\frac{E_k(v_1/v_2)}{(v_1/v_2)} \tag{4}$$

with v_2 of minimal degree.

Note that any RNF of $\mathscr{C}_k(T_2)$ has $v_2 \in \mathbb{F}[k]$ of the same minimal degree [9, Theorems 9,10].

An *additive decomposition* of $T(k)$ consists of a pair of similar hypergeometric terms (T_1, T_2) such that both Properties (i) and (ii) hold.

Lemma 1 (*Abramov and Perkovšek [6,9]*). *Let $T(k)$ be a hypergeometric term over $\mathbb{F}$ and (T_1, T_2) be an additive decomposition of $T(k)$. For any RNF of the form* (4) *of $\mathscr{C}_k(T_2)$, and for each irreducible $p \in \mathbb{F}[k]$ such that $p|v_2$, the following three properties hold*:

$$\begin{aligned}&\mathbf{Pa} : E_k^h p | v_2 \Rightarrow h = 0, \quad \mathbf{Pb} : E_k^h p | f_1 \Rightarrow h < 0 \quad \textit{and}\\ &\mathbf{Pc} : E_k^h p | f_2 \Rightarrow h > 0.\end{aligned} \tag{5}$$

If the hypergeometric term $T_2(k)$ in (3) is identically zero, then $T(k)$ is said to be k-*summable*. Otherwise, each irreducible factor p of v_2 has properties **Pa, Pb, Pc**, and T is k-non-summable.

Proposition 1 (*Abramov and Perkovšek [6,9]*). *Let an RNF of the k-certificate of a given hypergeometric term $T(k)$ be of the form* (4). *If there exists at least one irreducible factor p of v_2 such that all three properties* **Pa**, **Pb**, **Pc** *hold, then $T(k)$ is k-non-summable*.

Let $R(n, k) \in \mathbb{K}(n, k)$. By identifying the field $\mathbb{F}$ with $\mathbb{K}(n)$, the notion of an RNF of $R(n, k)$ w.r.t. k is well-defined. Let $T(n, k)$ be a bivariate hypergeometric term of n and k. Note that the algorithm which solves the additive decomposition problem only works with an RNF of the certificate R of T. By "an additive decomposition of $T(n, k)$ w.r.t. k", we identify the certificate R with $\mathscr{C}_k(T)$ and an RNF $F(E_k V)/V$ of $\mathscr{C}_k(T)$ is computed w.r.t. k. Additionally, T_1 and T_2 are hypergeometric terms of k, similar to T, i.e., there are $f_1, f_2 \in \mathbb{F}(k)$ such that $T_i = f_i\, T$. Since $\mathbb{F}(k) = \mathbb{K}(n)(k) = \mathbb{K}(n, k)$, both f_1 and f_2 are rational functions of n and k. Thus, T_i are rational-function (of n and k) multiples of T, and are hence hypergeometric terms of n and k.

Proposition 2. *For a hypergeometric term $T(n,k)$ of n and k, let $(T_1(n,k), T_2(n,k))$ be an additive decomposition of T w.r.t. k. Then*

(i) *a Z-pair for $T(n,k)$ exists if and only if a Z-pair for $T_2(n,k)$ exists*; *and*
(ii) *the minimal telescopers for T and T_2 are the same*.

Proof. (i) Let (L, G) be a Z-pair for T_2. It follows from (3) that $LT = (E_k - 1)(LT_1 + G)$. Since $T_1 \sim T_2$, $T_2 \sim G$, and $\sim$ is an equivalence relation, $LT_1 + G$ is a hypergeometric term [14, Proposition 5.6.2]. Consequently, $(L, LT_1 + G)$ is a Z-pair for T. On the other hand, let (L, G) be a Z-pair for T. By following the same argument, one can easily show that $(L, G - LT_1)$ is a Z-pair for T_2.

(ii) Let L be the minimal telescoper for T_2. It follows from (i) that L is a telescoper for T. Suppose there exists a telescoper $\tilde{L}$ for T and $\operatorname{ord} \tilde{L} < \operatorname{ord} L$. Then it follows from (i) that $\tilde{L}$ is a telescoper for T_2 and $\operatorname{ord} \tilde{L} < \operatorname{ord} L$. A contradiction. $\square$

Definition 2. A polynomial $p(n,k) \in \mathbb{K}[n,k]$ is *integer-linear* if it has the form

$$\alpha n + \beta k + \gamma, \quad \text{where} \quad \alpha, \beta \in \mathbb{Z} \text{ and } \gamma \in \mathbb{K}. \tag{6}$$

Theorem 1 (*Abramov and Perkovšek [8, Theorem 8]*). *For a hypergeometric term $T(n,k)$, let $F, V \in \mathbb{K}(n,k)$ be such that*

$$F \frac{E_k V}{V}$$

is an RNF over $\mathbb{K}(n)$ of $\mathscr{C}_k(T)$. Then there exists $D \in \mathbb{K}(n,k)$ so that $\mathscr{C}_n(T)$ can be written as

$$D \frac{E_n V}{V}, \quad D = \frac{d_1}{d_2}, \quad \gcd(d_1, d_2) = 1 \tag{7}$$

and the numerators and denominators of F and D all factor into integer-linear polynomials.

2.2. *The existence of a telescoper*

Recall that the fundamental theorem [15–17] provides only a sufficient condition for the termination of $\mathscr{Z}$. It states that a telescoper for a hypergeometric term $T(n,k)$ exists if $T(n,k)$ is *proper*, i.e., it can be written in the form

$$P(n,k) \frac{\prod_{i=1}^{l} \Gamma(p_i(n,k))}{\prod_{i=1}^{m} \Gamma(p'_i(n,k))} u^n v^k, \tag{8}$$

where $P(n,k) \in \mathbb{K}[n,k]$; $p_i(n,k)$, $p'_i(n,k)$ are integer-linear; $l, m \in \mathbb{N}$; $\mathbb{K}$ is a numeric field (e.g., $\mathbb{C}$); and $u, v \in \mathbb{K}$ and may contain parameters different from n and k.

It is well known that the set $\mathscr{S}$ of hypergeometric terms on which $\mathscr{Z}$ terminates is a proper subset of the set of all hypergeometric terms, but a proper super-set of the set of proper hypergeometric terms. The following theorem [1, Theorem 10] gives a complete description of $\mathscr{S}$. It provides a necessary and sufficient condition for the termination of $\mathscr{Z}$ on a hypergeometric term $T(n,k)$ (or equivalently, the applicability of $\mathscr{Z}$ to $T(n,k)$).

Theorem 2 (*Criterion for the existence of a telescoper*). *For a given hypergeometric term $T(n,k)$, let $(T_1(n,k),\ T_2(n,k))$ be an additive decomposition of T w.r.t. k. Let* (4) *be an RNF w.r.t. k over $\mathbb{K}(n)$ of the k-certificate of T_2. Then a telescoper for $T(n,k)$ exists if and only if each factor of $v_2(n,k)$ irreducible in $\mathbb{K}[n,k]$ is integer-linear.*

See [2, Section 5] for a description of the algorithm which determines the applicability of $\mathscr{Z}$ to a hypergeometric term $T(n,k)$. Note that the only information this algorithm needs is the k-certificate of T.

3. A lower bound for the order of telescopers for a minimal k-non-summable term

Definition 3. A *minimal k-non-summable* hypergeometric term $T(n,k)$ is a hypergeometric term where $\mathscr{C}_k(T)$ has an RNF w.r.t. k of the form (4), and for each irreducible p such that $p|v_2$, all three properties **Pa**, **Pb**, **Pc** hold.

For a given hypergeometric term $T(n,k)$, let $(T_1(n,k), T_2(n,k))$ be an additive decomposition of T w.r.t. k. It follows from Lemma 1 that T_2 is minimal k-non-summable. For the remainder of this section, we assume that $T(n,k)$ is minimal k-non-summable. Let us now introduce the notion of *crushing operators*.

Definition 4. Let $M \in \mathbb{K}[n, E_n]$ be such that $MT \neq 0$, and for any RNF w.r.t. k

$$F' \frac{E_k V'}{V'}, \quad V' = \frac{v_1'}{v_2'} \tag{9}$$

of $\mathscr{C}_k(MT)$, each of the irreducible factors of v_2' does not have at least one of the three properties **Pa**, **Pb**, **Pc**. Then M is a *crushing operator* for T.

Proposition 3. *If L is a telescoper for T, then L is a crushing operator for T.*

Proof. The claim follows from Proposition 1. □

Corollary 1. *If there does not exist any crushing operator for T of order less than μ, $\mu \geqslant 1$, then there does not exist any telescoper for T of order less than μ.*

Hence, the problem of computing a lower bound for the order of the telescopers for T is reduced to the problem of computing a lower bound for the order of a minimal crushing operator for T.

Theorem 3. *Let $F(E_kV)/V$ of the form* (4) *be an RNF w.r.t. k of $\mathscr{C}_k(T)$. Let $A = \mathscr{C}_n(T) = D(E_nV)/V$ be as defined in Theorem* 1. *Suppose that the polynomial $v_2 \in \mathbb{K}[n,k]$*

factors into integer-linear polynomials. Let $M \in \mathbb{K}[n, E_n]$ *be a crushing operator for* $T(n,k)$, $\operatorname{ord} M = \rho$. *Let p be an integer-linear factor of* v_2, $\deg_k p = 1$. *Then*

(i) *there exists an integer h such that*

$$E_k^h\ p | E_n v_2 \cdot E_n^2 v_2 \cdots E_n^{\rho} v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2; \text{ and} \tag{10}$$

(ii) *let* ρ_p *be the minimal value of* ρ *in* (i) *such that* (10) *is satisfied. Then the order of a minimal crushing operator for T is not less than* $\mu = \max_{p|v_2} \rho_p$.

Proof. (i) Let

$$M = a_\rho(n) E_n^\rho + \cdots + a_1(n) E_n + a_0(n), \quad a_i(n) \in \mathbb{K}[n].$$

Then

$$MT = \left(\sum_{m=0}^{\rho} a_m(n) A \cdot E_n A \cdots E_n^{m-1} A \right) T.$$

Therefore,

$$\mathscr{C}_k(MT) = F \frac{E_k R}{R}, \tag{11}$$

where

$$\begin{aligned} R &= V \sum_{m=0}^{\rho} a_m(n) A \cdot E_n A \cdots E_n^{m-1} A \\ &= V \sum_{m=0}^{\rho} a_m(n) \frac{E_n^m V}{V} D \cdot E_n D \cdots E_n^{m-1} D \\ &= \sum_{m=0}^{\rho} a_m(n) \frac{E_n^m v_1 \cdot d_1 \cdot E_n d_1 \cdots E_n^{m-1} d_1}{E_n^m v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{m-1} d_2}. \end{aligned}$$

Rewrite R as

$$R = \frac{r_1}{r_2}, \quad r_1,\ r_2 \in \mathbb{K}[n,k],$$

$$r_2 = v_2 \cdot E_n v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2, \quad r_1 = s_1 + v_2 s_2,$$

where s_2 is a polynomial from $\mathbb{K}[n,k]$, and $s_1 = a_0(n) \cdot E_n v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$.

If p is *not* a factor of the denominator r_2 of R, then since v_2 is a factor of r_2, p must divide the numerator r_1 of R, i.e.,

$$p | (s_1 + v_2 s_2).$$

Since p is a factor of v_2, this implies $p | s_1$. Additionally, p does not divide $a_0(n)$ since $\deg_k p = 1$. Therefore,

$$p | E_n v_2 \cdot E_n^2 v_2 \cdots E_n^\rho v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2. \tag{12}$$

If p is a factor of the denominator r_2, then since M is a crushing operator for T, at least one of the three properties **Pa**, **Pb**, **Pc** does not hold for p. Notice that $\mathscr{C}_k(T)$ in (4) and $\mathscr{C}_k(MT)$ in (11) have the same kernel F. It follows together with Lemma 1 that for the integer-linear factor p of v_2, properties **Pb** and **Pc** *always* hold. Consequently, property **Pa** does *not* hold, i.e., there exists an $h \in \mathbb{Z}\backslash\{0\}$ such that $E_k^h p$ divides r_2. Additionally, since T is minimal k-non-summable, it follows from property **Pa** that there does not exist an $h \in \mathbb{Z}\backslash\{0\}$ such that $E_k^h p | v_2$. This gives

$$E_k^h\, p | E_n v_2 \cdot E_n^2 v_2 \cdots E_n^{\rho} v_2 \cdot d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2. \tag{13}$$

It follows from (12) and (13) that (i) is satisfied.

(ii) The claim follows from the fact that for each factor p of v_2, there does not exist any crushing operator for T of order less than ρ_p. $\square$

It follows from Theorem 3 that if $\deg_k v_2 = 0$, then the computed lower bound is 1.

4. A general algorithm

For a given hypergeometric term $T(n, k)$ of n and k, an algorithm which computes a lower bound μ for the order of the telescopers for T consists of two steps. A check to determine the existence of a telescoper for T is performed in the first step. This is attained by first applying to $T(n, k)$ the algorithm which solves the additive decomposition problem w.r.t. k to construct two hypergeometric terms $T_1(n, k)$, $T_2(n, k)$ such that

$$T(n, k) = (E_k - 1)\, T_1(n, k) + T_2(n, k) \tag{14}$$

and $\mathscr{C}_k(T_2)$ has an RNF w.r.t. k of the form (4). If v_2 does not factor into integer-linear polynomials, then it follows from Theorem 2 that $\mathscr{Z}$ is not applicable to T, and there is no need to compute a lower bound μ. Otherwise, rewrite v_2 as a product of integer-linear polynomials each of which is of the form (6). An algorithm, based on gcd and resultant computation, for verifying if $v_2 \in \mathbb{K}[n, k]$ factors into integer-linear polynomials, and if this is the case, rewrite v_2 in the desired factored form as described in [3,5]. Without loss of generality, we can assume that $\gcd(\alpha, \beta) = 1$, and $\beta \geqslant 0$.

In the second step, since $\mathscr{Z}$ is applicable to T, if follows from Proposition 3 that the existence of the crushing operators for T_2 is guaranteed. Additionally, all the hypotheses required for computing a lower bound μ for the order of the telescopers for T_2 exist. More precisely, one can apply Theorem 3 to T_2 to compute a lower bound μ. It follows from Proposition 2 that one can use μ as a lower bound for the order of the telescopers for T.

For each integer-linear factor p of v_2, $\deg_k p = 1$, the second step requires the computation of the minimal value of ρ in the pair (ρ, h), $h \in \mathbb{Z}$, $\rho \in \mathbb{N}\backslash\{0\}$ such that

(i) $E_k^h p | E_n v_2 \cdot E_n^2 v_2 \cdots E_n^{\rho} v_2$ or
(ii) $E_k^h p | d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$.

Consider the following simple algorithm $C_{(\mathrm{i})}$:

`algorithm` $C_{(\mathrm{i})}$
`input`: $p = \alpha n + \beta k + \gamma,\ \alpha,\ \beta \in \mathbb{Z},\ \gcd(\alpha, \beta) = 1,\ \beta > 0,\ \gamma \in \mathbb{K},$
$v_2 = \prod_{i=1}^{m}(\alpha_i n + \beta_i k + \gamma_i),\ \alpha_i,\ \beta_i \in \mathbb{Z},\ \gcd(\alpha_i, \beta_i) = 1,\ \beta_i \geqslant 0,\ \gamma_i \in \mathbb{K};$
`output`: the minimal value of $\rho \in \mathbb{N}\backslash\{0\}$ such that (i) is satisfied;

$\rho_{\min} := \infty$;
for $i = 1, 2, \ldots, m$ **do**
  if $\alpha = \alpha_i$ **and** $\beta = \beta_i$ **and** $\gamma - \gamma_i \in \mathbb{Z}$ **then**
    find the minimal $\rho \in \mathbb{N}\backslash\{0\}$ and $h \in \mathbb{Z}$ such that
      $\alpha\rho - \beta h = \gamma - \gamma_i$;
    $\rho_{\min} := \min\{\rho_{\min}, \rho\}$
  fi
od;
return $\rho_{\min}$.

For a given integer-linear factor p of v_2, $\deg_k p = 1$, the algorithm $C_{(\mathrm{i})}$ simply iterates through each integer-linear polynomial q of v_2. If $p - q = \sigma \in \mathbb{Z}$, then the algorithm solves the diophantine equation $\alpha\rho - \beta h = \sigma$, and chooses the minimal positive value of ρ. (Note that since $\gcd(\alpha, \beta) = 1$, the solution is guaranteed to exist.)

An algorithm $C_{(\mathrm{ii})}$ which finds the minimal value of ρ such that (ii) is satisfied can be described in a very similar manner. Note that it follows from Theorem 1 that the polynomial $d_2 \in \mathbb{K}[n, k]$ in (7) factors into integer-linear polynomials.

By iterating through each factor p of v_2, we obtain the desired lower bound μ. This leads to the following algorithm which computes in many examples (see below) convincing lower bounds for the minimal orders of the telescopers for hypergeometric terms.

`algorithm` *LowerBound*;
`input`: a hypergeometric term $T(n, k)$;
`output`: a lower bound μ for the order of the telescopers for T;

  apply the algorithm which solves the additive decomposition
    problem w.r.t. k to obtain $T_1(n, k)$, $T_2(n, k)$ in (14);
  if $T_2 = 0$ **then return** 0 **fi**;
  at this point, $\mathscr{C}_k(T_2)$ has an RNF w.r.t. k of the form (4);
  if the polynomial $v_2(n, k)$ in (4) is written as
    $v_2 = \prod_{i=1}^{s} p_i$, where $p_i = (\alpha_i n + \beta_i k + \gamma_i)$,
    $\alpha_i,\ \beta_i \in \mathbb{Z},\ \gcd(\alpha_i, \beta_i) = 1,\ \beta_i \geqslant 0,\ \gamma_i \in \mathbb{K}$ **then**
    if $s = 0$ **then return** 1 **fi**;
    $\mu := -\infty$;
    $d_2 :=$ denominator$(\mathscr{C}_n(T)(v_1/v_2)/E_n(v_1/v_2))$;
    Rewrite d_2 as $\prod_{j=1}^{t} q_j$, where $q_j = (\alpha_j n + \beta_j k + \gamma_j)$,
      $\alpha_j,\ \beta_j \in \mathbb{Z},\ \gcd(\alpha_j, \beta_j) = 1,\ \beta_j \geqslant 0,\ \gamma_j \in \mathbb{K}$;
    for $i = 1, 2, \ldots, s$ **do**

if $\deg_k p_i = 1$ **then**
$\mu_{\min} := C_{(\mathrm{i})}(p_i, v_2)$;
$\mu_{\min} := \min\{\mu_{\min}, C_{(\mathrm{ii})}(p_i, d_2)\}$;
$\mu := \max\{\mu, \mu_{\min}\}$
fi
od;
return μ
else
return "Zeilberger's algorithm is not applicable"
fi;

Note that instead of rewriting d_2 as a product of integer-linear polynomials, and using it in the call $C_{(ii)}(p_i, d_2)$ in `LowerBound`, it is possible to use a simpler polynomial which is a divisor of d_2. For a given $f \in \mathbb{K}[n, k]$ and $c \in \mathbb{Q}$, there exists an algorithm [5] (called wc) which extracts the maximal factor $w \in \mathbb{K}[n, k]$ from f where w can be written in the form

$$\prod_i (k + c\,n + \gamma_i), \quad \gamma_i \in \mathbb{K}.$$

Hence, for each factor $p = (\alpha n + \beta k + \gamma)$ of v_2, we call wc with d_2 and α/β as input. This helps to reduce the number of integer-linear factors of d_2 to be compared with p.

Example 1. Consider the hypergeometric term

$$T = \frac{1}{(5n + 2k + 1)(-3n + 5k + 5)}.$$

(T is also a rational function of n and k.) Applying the algorithm which solves the additive decomposition problem yields two hypergeometric terms $T_1(n, k) = 0$ and $T_2(n, k) = T(n, k)$ in (14). Since T is a rational function, the polynomial v_2 in (4), and subsequently d_2 in (7) can be readily rewritten as

$$v_2 = (5n + 2k + 1)(-3n + 5k + 5), \quad d_2 = 1.$$

Since v_2 can be written as a product of integer-linear polynomials, it follows from Theorem 2 that $\mathscr{Z}$ is applicable to T, and the two possible values for the integer-linear factor p are

$$p_1 = 5n + 2k + 1, \quad p_2 = -3n + 5k + 5.$$

When $p = p_1 = 5n + 2k + 1$, the diophantine equation to be solved is $5\rho - 2h = 0$, which yields $(\rho_1, h_1) = (2, 5)$ as the solution. When $p = p_2 = -3n + 5k + 5$, the diophantine equation to be solved is $-3\rho - 5h = 0$, which yields $(\rho_2, h_2) = (5, -3)$ as the solution. Therefore, a lower bound μ for the order of the telescopers for T is $\mu = \max\{2, 5\} = 5$. Note that invoking $\mathscr{Z}$ on T results in the minimal telescoper L of order 6 where

$$L = (31n + 181)E_n^6 + (31n + 150)E_n^5 - (31n + 26)E_n - (31n - 5).$$

Example 2. Consider the class of hypergeometric terms of the form

$$T = \frac{1}{(a_1 n + b_1 k + c_1)\,(a_2 n + b_2 k + c_2)!}, \tag{15}$$

where $a_1,\ b_1,\ a_2,\ b_2 \in \mathbb{Z},\ \gcd(a_1, b_1) = 1,\ b_1 \neq 0,\ a_1 \neq a_2$ or $b_1 \neq b_2$. Without loss of generality, we can assume that $b_1 > 0$. Applying the algorithm which solves the additive decomposition problem yields two hypergeometric terms $T_1(n,k)=0$ and $T_2(n,k)=T(n,k)$ in (14), and the polynomial v_2 in (4) is

$$a_1 n + b_1 k + c_1,$$

which is also the only possible value of p. Subsequently, the value of d_2 in (7) is

$$\begin{array}{ll} d_2 = (a_2 n + b_2 k + c_2 + 1) \cdots (a_2 n + b_2 k + a_2 + c_2) & \text{if } \ a_2 > 0, \\ d_2 = 1 & \text{if } \ a_2 = 0, \\ d_2 = (a_2 n + b_2 k + c_2 + a_2 + 1) \cdots (a_2 n + b_2 k + c_2) & \text{if } \ a_2 < 0. \end{array}$$

Since $a_1 \neq a_2$ or $b_1 \neq b_2$, there does not exist any integer h such that $E_k^h p | d_2 \cdot E_n d_2 \cdots E_n^{\rho-1} d_2$. When $p = a_1 n + b_1 k + c_1$, the diophantine equation to be solved is $a_1 \rho - b_1 h = 0$, which yields $(\rho_1, h_1) = (b_1, a_1)$ as the solution. Therefore, a lower bound μ for the order of the telescopers for T is $\mu = b_1$.

In summary, for the class of hypergeometric terms of the form (15), the polynomial factor $(a_1 n + b_1 k + c_1)$ is the *dominant* factor. It determines the lower bound (which is b_1) for the order of the minimal telescoper for T. As an example, the computed lower bound for the minimal telescoper for

$$T = \frac{1}{(n - 9k - 2)(2n + k + 3)!}$$

is 9, while the order of the minimal telescoper for T is 10. By first computing this lower bound, we can safely avoid the computation of a telescoper of order less than 9 (in addition to the assurance that the telescopers for T do exist). On the other hand, if $b_1 = 1$, then the computed lower bound μ equals 1, i.e., the lowest possible value for μ. As an example, the computed lower bound for the minimal telescoper for

$$T = \frac{1}{(n + k + 1)(n + 5k + 2)!}$$

is 1, while the order of the minimal telescoper for T is 6.

Note that when the factorial term $(a_2 n + b_2 k + c_2)!$ in (15) equals 1, we have b_1 as a lower bound for the order of the minimal telescoper for T. This lower bound also equals the order of the minimal telescoper for T (see [10]).

5. Implementation

The algorithm which computes a lower bound for the order of the telescopers and related functions are implemented in the computer algebra system Maple [12]. The Maple source code, and test results reported in this paper are available, and can be downloaded from `http://www.scg.uwaterloo.ca/~hqle/code/LowerBound/LB.html`. These functions include

1. `AdditiveDecomposition` solves the additive decomposition problem;
2. `IsZApplicable` determines the applicability of Zeilberger's algorithm;
3. `Zeilberger` computes the minimal *Z*-pair of the given hypergeometric term; and
4. `LowerBound` computes a lower bound for the order of the telescopers.

The function `LowerBound` has the calling sequence

$$\text{LowerBound}(T, n, k, E_n, Zpair);$$

where *T* is a hypergeometric term of *n* and *k*, and E_n denotes the shift operator w.r.t. n. (E_n and *Zpair* are optional arguments). If the non-existence of a *Z*-pair (L, G) for *T* is guaranteed, then `LowerBound` returns the conclusive error message "Zeilberger's algorithm is not applicable." Otherwise, the output is a non-negative integer μ denoting the value of the computed lower bound for the order of L. In this case, if the optional arguments E_n and *Zpair* (each of which can be any unassigned name) are given, then the function `Zeilberger` is invoked starting with μ as a lower bound for the order of L, and *Zpair* will be assigned to the computed *Z*-pair (L, G).

Note that there are different Maple implementations of $\mathscr{Z}$ such as `zeil` in the EKHAD package [14], and `sumrecursion` in the `sumtools` package. A Mathematica implementation is presented in [13]. Since the terminating condition that allows a hypergeometric term to have a *Z*-pair is unknown at the time these functions were implemented, an upper bound for the order of the recurrence operator *L* in the *Z*-pair (L, G) needs to be specified in advance (for instance, the default values are 6 for the parameter `MAXORDER` in `zeil`, and 5 for the global parameter '`sum/zborder`' in `sumrecursion`). As a consequence, when given a hypergeometric term $T(n, k)$ as input, (1) these programs might fail even if a *Z*-pair exists, i.e., the maximum order of *L* is not set "high enough", or (2) they simply "waste" CPU time trying to find a *Z*-pair when no such *Z*-pair exists. The function `LowerBound`, on the other hand, first determines the applicability of $\mathscr{Z}$ to $T(n, k)$. If the existence of a *Z*-pair is guaranteed, then it computes a lower bound μ for the order of L, and if requested, calls $\mathscr{Z}$ using μ as the starting value for the order of L, instead of 0. Since the existence of a *Z*-pair is guaranteed, there is no need to set an upper bound for the order of L.

The remainder of the paper is devoted to various experiments. For an input hypergeometric term $T(n, k)$ with an additive decomposition $(T_1(n, k), T_2(n, k))$. Let μ and ρ be the computed lower bound and the order of the minimal telescoper for *T*, respectively. The results show that

1. the time to compute a lower bound, including the time to determine whether $\mathscr{Z}$ is applicable to *T*, is negligible in comparison with the time to compute telescopers of order less than μ; and
2. for the case where $T_1 \neq 0$, since T_2 is simpler than *T* in some sense, some speed-up can be obtained if we first compute the minimal *Z*-pair (L, G) for T_2. It follows from Proposition 2 that $(L, LT_1 + G)$ is the minimal *Z*-pair for *T*.

Table 1
Example 3—time and space requirement

μ	ρ	t_1	t_2	m_1	m_2
8	8	0.28	4.17	3,286	60,123

Example 3. Consider the hypergeometric term

$$T(n,k) = \frac{1}{(2k-1)(n-8k+1)} \binom{2n-2k}{n-k} \binom{2k}{k}.$$

The computed lower bound μ is 8 which equals the order ρ of the minimal telescoper for T. Let $t_1,\ m_1$ denote the time (in seconds) and memory (in kilobytes) required to compute a lower bound μ, and t_2 and m_2 denote the (wasted) time and memory required to compute telescopers of order less than μ. Table 1 shows the figures for $t_i, m_i, 1 \leqslant i \leqslant 2$ for the given T.[1]

It takes 11.84 s and 6.96 s to compute the minimal Z-pair for T using 0 and 8 as the starting values of the guessed order for the telescopers, respectively. Note that if one applies `Zeilberger` directly to T, one needs to set an upper bound for the telescopers to a high enough value. For instance, if it is set to 7 in this example, then the function will return the inconclusive message:

Error, (in Zeilberger) No recurrence of order 7 was found

Example 4. Consider the hypergeometric term

$$T(n,k) = \frac{1}{nk+1} \binom{2n}{2k}.$$

It takes `LowerBound` 0.23 s and 3,047 kilobytes to return the error message "Error, (in LowerBound) Zeilberger's algorithm is not applicable". The function recognizes that the polynomial $v_2(n,k)$ in (4) is $(nk+1)$ which does not factor into a product of integer-linear polynomials, and returns the conclusive answer quickly. On the other hand, it takes `Zeilberger` 12.15 s and 175,401 kilobytes to return the error message "Error, (in Zeilberger) No recurrence of order 6 was found". The function does not know whether a Z-pair (L, G) for T exists. It tries to compute one and returns the above inconclusive answer. Since there does not exist a Z-pair for T, the higher the value of the upper bound for the order of L set, the more the time and memory wasted (see Table 2).

Example 5. In this example, we randomly generated a set of 10 hypergeometric terms each of which is of the form

$$T(n,k) = \frac{1}{(a_1 n + b_1 k + c_1)(a_2 n + b_2 k + c_2)!}, \quad a_i, b_i, c_i \neq 0,$$
$$-3 \leqslant a_i, b_i, c_i \leqslant 3, \ -10 \leqslant b_1 \leqslant 10, \ -2 \leqslant b_2 \leqslant 2.$$

[1] All the reported timings were obtained on a 1 GHz Compaq Deskpro Workstation with 512 Mb RAM.

Table 2
Example 4—$\mathscr{Z}$ is not applicable to the input hypergeometric term

Upper bound	Wasted time
6	12.15
8	179.03
10	1,605.73

Table 3
Example 5—time and space requirement

i	μ	ρ	t_1	t_2	m_1	m_2	`Lb`	`Zb`
1	10	11	0.09	4.79	1,661	61,935	11.72	17.22
2	10	11	0.08	13.87	896	185,289	32.72	45.25
3	9	10	0.15	7.00	1,200	94,735	16.73	22.42
4	9	11	0.20	9.59	1,519	117,734	67.77	72.50
5	8	9	0.06	1.62	770	17,712	2.82	4.41
6	8	9	0.09	9.29	1,027	123,202	33.80	40.91
7	9	10	0.06	3.02	965	35,203	6.77	10.02
8	9	10	0.08	8.95	993	121,058	25.49	33.86
9	7	8	0.15	4.68	1,132	59,468	13.36	17.51
10	10	11	0.14	18.87	935	244,346	62.31	75.14
Total			1.10	81.68	11,098	1,060,682	273.49	339.24

Table 3shows a comparison similar to that of Table 1 in Example 3. Additionally, we also added the time to compute the minimal Z-pair using `Zeilberger` (`Zb`) and `LowerBound` (`Lb`).

Example 6. For a given hypergeometric term $T(n,k)$, let $(T_1(n,k), T_2(n,k))$ be an additive decomposition of T w.r.t. k. If $T_1 \neq 0$, instead of applying $\mathscr{Z}$ to T, we suggest that $\mathscr{Z}$ be applied to T_2. Following Proposition 2, the required minimal Z-pair for $T(n,k)$ can then be easily obtained from the computed minimal Z-pair for $T_2(n,k)$. This in general helps to reduce the size of the problem to be solved. As an example, for $b \in \mathbb{N}\backslash\{0\}$, $j \in \{1,3\}$, let

$$T_1(n,k) = \frac{1}{(nk-1)(n-bk-2)^j(2n+k+3)!},$$
$$T_2(n,k) = \frac{1}{(n-bk-2)(2n+k+3)!}.$$

Consider

$$T(n,k) = (E_k - 1)\,T_1(n,k) + T_2(n,k).$$

Since $T_1 \sim T_2$, T is a hypergeometric term, let t_1 be the time to compute a lower bound μ (which is b by Example 2) and t_2, t_3 be the times to compute the minimal Z-pair for T by applying $\mathscr{Z}$ to T_2 and T, respectively, using μ as the starting value for the guessed order

Table 4
Example 6—timing comparison

j	b	Timing (seconds)		
		t_1	t_2	t_3
	1	1.03	0.51	1.55
	2	1.09	3.99	9.30
1	3	1.09	5.00	35.32
	4	1.15	7.01	130.45
	5	1.09	10.03	2320.07
	1	2.58	2.64	4.83
	2	2.79	27.71	53.67
3	3	2.93	34.44	264.69
	4	2.81	34.22	1,675.19
	5	2.92	42.55	19,301.48

of the telescopers. Table 4 shows the timing comparison. One can easily notice that as b and/or j increase, the relative performance of `Zeilberger` (compared to `LowerBound`) quickly worsens.

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