On a Set of Hyperexponential Elements and the Fast Versions of Zeilberger's Algorithm

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ABSTRACT
Let T be a bivariate hyperexponential element for which the existence of a Z-pair is guaranteed. We define a set $\mathcal{S}$ of hyperexponential elements, show that the fast versions of Zeilberger's algorithm are not applicable to T if and only if T is in $\mathcal{S}$, and provide a direct algorithm for computing the minimal Z-pair for T in $\mathcal{S}$.

1. INTRODUCTION
For a given bivariate hyperexponential $T(x, y)$, Zeilberger's algorithm [16, 18, 19, 20, 3, 10] *tries* to construct for T a *Z-pair* (L, G), which consists of a linear differential or difference operator L with coefficients in $\mathbb{C}(x)$, and a bivariate hyperexponential

$$G(x, y) = R(x, y)\, T(x, y),\ \ R(x, y) \in \mathbb{C}(x, y). \tag{1}$$

The constructed *telescoper* L in the returned Z-pair is of minimal possible order, and is called *the minimal telescoper.* We name the Z-pair (L, G), where L is the minimal telescoper, the *minimal Z-pair.*

There exist two versions of Zeilberger's algorithm. The first one [19], also known as the *"slow"* version, is based on the holonomic systems approach. The second one [20, 3, 18], also known as the *"fast"* version or *the method of creative telescoping*, is based on Gosper's algorithm [8]. Needless to say, all known implementations of Zeilberger's algorithm are based on the fast version, named hereafter as $\mathcal{Z}$ for short.

For a hyperexponential $T(x, y)$, the statement "$\mathcal{Z}$ is applicable to T" means that $\mathcal{Z}$ terminates given T as input, and succeeds in computing the minimal Z-pair for T. If there exists a Z-pair for T, then it is widely agreed that $\mathcal{Z}$ is applicable to T. That is, the two statements (A) "There exists a Z-pair for T", and (B) "$\mathcal{Z}$ is applicable to T" are equivalent. However, for the hypergeometric term

$$T_1(n, k) = (3nk^2 + (n^2 + 1)k + n^2 + 1) \binom{2k}{k}^3, \tag{2}$$

the q-hypergeometric term

$$T_2(q^n, q^k) = q^{n+k} \binom{2\,k}{k}_q, \tag{3}$$

and the exponential function $T_3(x, y)$

$$(y^3 - (x-2)y^2 - (x-1)y) \exp\left(\frac{(x-1)^2}{x} + \frac{(y+2)^3}{y}\right), \tag{4}$$

experimentation shows that $\mathcal{Z}$ does not seem to terminate on T_1, T_2, and T_3, even though the minimal Z-pairs for T_1, T_2 and T_3 exist and can be computed (Examples 5, 6, 7).

Let $T(x, y)$ be a hyperexponential element and the existence of a Z-pair for T is guaranteed. We define a set $\mathcal{S}$ of hyperexponential elements, and show that $\mathcal{Z}$ is not applicable to T if and only if $T \in \mathcal{S}$. As a consequence, the two statements (A) and (B), strictly speaking, are not equivalent. We provide a direct algorithm for handling elements of $\mathcal{S}$. With the incorporation of this algorithm into $\mathcal{Z}$, the two statements (A) and (B) become equivalent.

This paper is organized as follows: Section 2 reviews a ring of Ore polynomials suitable to describe linear differential and difference operators in a uniform manner. Section 3 describes Gosper's algorithm and Zeilberger's algorithm in the setting of Ore polynomials. The set $\mathcal{S}$ is defined in Section 4. Section 5 provides an algorithm for recognizing if a hyperexponential element T is in $\mathcal{S}$, and a direct algorithm for computing the minimal Z-pair for T in $\mathcal{S}$.

2. ORE POLYNOMIAL RINGS AND HYPEREXPONENTIAL ELEMENTS

Ore introduces a type of noncommutative rings of univariate polynomials to describe linear ordinary differential and difference operators [13]. These rings are referred as Ore rings in literatures. Chyzak and Salvy extend the notion of Ore rings to the multivariate case and put the respective elimination processes for linear partial differential and difference operators in a uniform setting [7]. In Gosper's algorithm and Zeilberger's algorithm, linear partial differential and difference operators have rational function coefficients, and act on multivariate functions. We describe these algorithms precisely and uniformly by orthogonal Ore rings [11], which is a special type of general multivariate Ore rings.

Let k be a commutative field of characteristic zero, and σ_i be an automorphism of k for $i = 1, \ldots, n$.

Definition 1. *A σ_i-derivation is an additive mapping δ_i from k to itself satisfying*

$$\delta_i(ab) = \sigma_i(a)\,\delta_i(b) + \delta_i(a)\,b \quad \text{for all } a, b \in k. \tag{5}$$

Let δ_i be a σ_i-derivation for $i = 1, \ldots, n$. We define the set $\Phi = \{(\sigma_1, \delta_1), \ldots, (\sigma_n, \delta_n)\}$ to be *commutative* if for all $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$:

$$\sigma_i \circ \sigma_j = \sigma_j \circ \sigma_i,\ \ \sigma_i \circ \delta_j = \delta_j \circ \sigma_i,\ \ \delta_i \circ \delta_j = \delta_j \circ \delta_i. \tag{6}$$

The set of constants with respect to σ_i and δ_i is

$$\mathrm{Const}_{\sigma_i,\delta_i}(k) = \{a \in k \,:\, \sigma_i(a) = a,\ \delta_i(a) = 0\}.$$

An element c is a *constant* if c is a constant with respect to all σ_i and δ_i, for $1 \leq i \leq n$.

Definition 2. *An Ore polynomial ring over k, given by a commutative set Φ, and denoted by*

$$\mathbb{A}_\Phi = k[\partial_1, \ldots, \partial_n; (\sigma_1, \delta_1), \ldots, (\sigma_n, \delta_n)],$$

is the ring of polynomials in ∂_i over k with the usual polynomial addition, and with the multiplicative rules given by

$$\partial_i\,\partial_j = \partial_j\,\partial_i,\ \ \partial_i(a) = \sigma_i(a)\,\partial_i + \delta_i(a) \quad \text{for all } a \in k.$$

The existence of Ore polynomial rings is verified in [7]. If c is a constant with respect to σ_i and δ_i, then $\partial_i c = c\partial_i$. Hence, c is also called a ∂_i-constant. Elements of the ring $\mathbb{A}_\Phi$ are called Ore polynomials.

Example 1. For any differential field k with derivation δ, $k[D; (1_k, \delta)]$ is the ring of linear ordinary differential operators. If $k = \mathbb{C}(n)$ and σ is the automorphism of k over $\mathbb{C}$ that takes n to $n+1$, then $k[E_n; (\sigma, 0)]$ is the ring of linear shift operators. If $k = \mathbb{C}(q)(t)$ and σ is the automorphism of k over $\mathbb{C}(q)$ that takes t to qt, then $k[Q_t; (\sigma, 0)]$ is the ring of linear ordinary q-shift operators.

For a uniform description of differential $\mathcal{Z}$ and difference $\mathcal{Z}$, we specialize the base field k to $\mathbb{C}(x_1, \ldots, x_n)$, which is denoted by $\mathbb{F}$. We use $\mathbf{1}$ to denote the idenity mapping and $\mathbf{0}$ the mapping that sends everything to zero, respectively.

Definition 3. *The commutative set Φ is said to be* orthogonal *if, for all i with $1 \leq i \leq n$, and $f \in \mathbb{F}$,*

1. *$\delta_i \neq \mathbf{0}$ if $\sigma_i = \mathbf{1}$;*
2. *$\delta_i(x_i) \in \mathbb{C}(x_i)$ and $\sigma_i(x_i) \in \mathbb{C}[x_i]$;*
3. *If $\delta_i \neq \mathbf{0}$, then*
$$\delta_i(f) = 0 \iff f \in \mathbb{C}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n);$$
4. *If $\sigma_i \neq \mathbf{1}$, then*
$$\sigma_i(f) = f \iff f \in \mathbb{C}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$

If Φ is orthogonal, $\mathbb{A}_\Phi$ is said to be an orthogonal Ore ring.

The first requirement in Definition 3 means that $\mathbb{F}$ is not a subset of $\mathrm{Const}_{\sigma_i,\delta_i}(\mathbb{F})$. The last two mean that

$$\mathrm{Const}_{\sigma_i,\delta_i}(\mathbb{F}) = \mathbb{C}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$$

for every i with $1 \leq i \leq n$.

Theorem 1 in [11] shows that an orthogonal Ore ring is isomorphic to $\mathbb{A}_\Delta$, where the commutative and orthogonal set

$$\Delta = \{(\sigma_1, \delta_1), \ldots, (\sigma_n, \delta_n)\}, \tag{7}$$

in which either $\sigma_i = \mathbf{1}$ or $\delta_i = \mathbf{0}$ for all i with $1 \leq i \leq n$. Instances of orthogonal Ore rings include the rings of partial differential operators, partial difference operators, or any mixture thereof.

Now, we define a *commutative* ring on which Ore polynomials in $\mathbb{A}_\Delta$ will act.

Definition 4. *A Δ-ring $\mathbb{E}$ is an $\mathbb{F}$-algebra to which the σ_i's and δ_i's in (7) can be extended in such a way that*

1. *all the σ_i are monomorphisms of $\mathbb{E}$;*
2. *if σ_i is $\mathbf{1}$ on $\mathbb{F}$, so is σ_i on $\mathbb{E}$;*
3. *all the δ_i's are derivations on $\mathbb{E}$;*
4. *if δ_i is $\mathbf{0}$ on $\mathbb{F}$, so is δ_i on $\mathbb{E}$.*

For $\partial_i \in \mathbb{A}_\Delta$ and an element a in a Δ-ring $\mathbb{E}$, the action of ∂_i on a is defined to be $\sigma_i(a)$ if $\delta_i = \mathbf{0}$, and $\delta_i(a)$ if $\sigma_i = \mathbf{1}$. This action is denoted by $\partial_i \bullet a$. For $a \in \mathbb{F}$, $\partial_i a$ means the product of ∂_i and a in $\mathbb{A}_\Delta$. This product is equal to $(\sigma_i(a)\partial_i + \delta_i(a))$, which is different from $\partial_i \bullet a$. As described in Section 4 of [11], the actions of the ∂_i's induce a well-defined action of Ore polynomials in $\mathbb{A}_\Delta$ on $\mathbb{E}$. This action is linear over the ring of constants of $\mathbb{E}$. It is also shown in Section 4 of [11] that ∂_i is a pseudo-linear operator, that is,

$$\partial_i \bullet (ab) = \sigma_i(a)\partial_i \bullet b + \delta_i(a)b \quad \text{for all } a, b \in \mathbb{E}. \tag{8}$$

An element c of $\mathbb{E}$ is said to be a constant with respect to ∂_i if $\sigma_i(c) = c$ and $\delta_i(c) = 0$. It follows from the pseudo-linearity of ∂_i that $(\partial_i \bullet (ca)) = c(\partial_i \bullet a)$ if c is a ∂_i-constant. An element of $\mathbb{E}$ is a constant if it is a constant with respect to all the ∂_i's.

The input of Gosper's algorithm and Zeilberger's algorithm is an exponential function in the differential case, and a hypergeometric term in the difference case. Below is a uniform description of both notions.

DEFINITION 5. *A nonzero element T of a Δ-ring is* hyperexponential *in x_i over $\mathbb{F}$ if $\partial_i \bullet T = u_i T$ with $u_i \in \mathbb{F}$. The element u_i is called the ∂_i-certificate of T. A nonzero element T is hyperexponential over $\mathbb{F}$ if T is hyperexponential in x_i over $\mathbb{F}$ for all i with $1 \le i \le n$.*

A hyperexponential element is called a hypergeometric term in the shift case, a q-hypergeometric term in the q-shift case, and an exponential function in the differential case.

A straightforward calculation shows

LEMMA 1. *If T is a hyperexponential element and L is in $\mathbb{A}_\Delta$, then $L \bullet T = RT$ for some $R \in \mathbb{F}$.*

REMARK 1. Lemma 1 does not assert that $(L \bullet T)$ is hyperexponential, because the rational function R may be zero.

It is shown in [17] that, in general, hyperexponential elements are not contained in a field. This motivates us to define them in a ring. As we do not assume hyperexponential elements are invertible, their properties will be proved by the compatibility of their certificates, as defined below.

DEFINITION 6. *Elements $P_1, \ldots P_n$ in $\mathbb{F}$ are compatible if, for all $1 \le i < j \le n$,*

$$P_i\, \sigma_i(P_j) + \delta_i(P_j) = P_j\, \sigma_j(P_i) + \delta_j(P_i). \tag{9}$$

The commutativity of ∂_i and ∂_j implies

LEMMA 2. *If T is a hyperexponential element, then their certificates are compatible.*

3. GOSPER'S ALGORITHM AND ZEILBERGER'S ALGORITHM

This section reviews Gosper's and Zeilberger's algorithms. For notational convenience we set $\delta_{x_i}{=}\delta_i$ and $\sigma_{x_i}{=}\sigma_i$.

3.1 Gosper's algorithm

Let y be one of the x_i's. Set

$$\nabla_y = \begin{cases} \partial_y & \text{if } \sigma_y = \mathbf{1}, \\ \partial_y - \mathbf{1} & \text{if } \sigma_y \neq \mathbf{1}. \end{cases}$$

Given a hyperexponential T in y, Gosper's algorithm determines if there exists a hyperexponential G in y such that

$$T = \nabla_y \bullet G, \tag{10}$$

and computes such a G provided that it exists. In this case T is said to be *hyperexponential integrable with respect to y*.

For a hyperexponential G that satisfies (10), there is a function $z \in \mathbb{F}$ such that $G = zT$, which, together with (10) and $\partial_y \bullet T = rT$ for some $r \in \mathbb{F}$, implies

$$\begin{cases} \delta_y(z) + r\, z = 1 & \text{if } \sigma_y = 1, \\ r\, \sigma_y(z) - z = 1 & \text{if } \sigma_y \neq 1. \end{cases} \tag{11}$$

The problem of finding a hyperexponential solution G of (10) is reduced to that of finding a rational solution z of (11).

REMARK 2. For the shift case, equation (11) has the form $r(y)\, z(y+1) - z(y) = 1$; for the q-shift case, it has the form $r(y)\, z(qy) - z(y) = 1$; and for the differential case, it has the form $z'(y) + r(y)z(y) = 1$. These forms are given in descriptions of Gosper's algorithms [20, 10, 3, 16, 14].

Let $\mathbb{K}$ be the subfield of $\mathbb{F}$ consisting of elements free of y.

- (q-)shift: For $r \in \mathbb{F}$, there are $(a, b, c) \in \mathbb{K}[y]^3$ such that (i) $r = a/b \times \sigma_y(c)/c$, (ii) $\gcd(a, \sigma_y^h(b)) = 1$ for all $h \in \mathbb{N}$. Then the problem of finding a rational solution $r\, \sigma_y(z) - z = 1$ in (11) is reduced to finding a polynomial solution $w(y)$ of
$$a(y)\, \sigma_y(w(y)) - \sigma_y^{-1}(b(y))\, w(y) = c(y); \tag{12}$$
- differential: For $r \in \mathbb{F}$, there are $(a, b, c) \in \mathbb{K}[y]^3$ such that (i) $r = a/b + \delta_y(c)/c$, (ii) $\gcd(b, a - h\delta_y(b)) = 1$ for all $h \in \mathbb{N}$. Then the problem of finding a rational solution $\delta_y(z) + r\, z = 1$ in (11) is reduced to finding a polynomial solution $w(y)$ of
$$b(y)\, \delta_y(w(y)) + (a + \delta_y(b(y)))\, w(y) = c(y). \tag{13}$$

A polynomial solution of either (12) or (13) can be computed by the method of undetermined coefficients. First one computes an upper bound d for the degree of the polynomial $w(y)$ (see [8] for the shift case, [10] for the q-shift case, and [3] for the differential case). The value of d is determined by a, b, and $\deg_y c$. Then one substitutes a generic polynomial of degree d for $w(y)$ into (12) or (13), equates the coefficients of like powers in y. This results in a linear system of algebraic equation, and the problem is reduced to determining if this system is consistent or not.

3.2 Zeilberger's algorithm

Let x and y be two distinct elements of $\{x_1, \ldots, x_n\}$, and $\mathbb{K}$ be the field of rational functions in $(\{x_1, \ldots, x_n\} \setminus \{x, y\})$ over $\mathbb{C}$. For a given hyperexponential T in x and y, $\mathcal{Z}$ tries to construct for T a Z-pair (L, G) which consists of a linear operator L with coefficients in $\mathbb{K}(x)$

$$L = a_\rho \partial_x^\rho + \cdots + a_1 \partial_x + a_0,\ a_\rho \neq 0,\ a_i \in \mathbb{K}(x), \tag{14}$$

and a hyperexponential G of the form

$$G = RT,\ R \in \mathbb{F} \tag{15}$$

such that

$$L \bullet T = \nabla_y \bullet G.$$

$\mathcal{Z}$ uses an *item-by-item examination* on the order ρ of the operator L in (14). It starts with the value of 0 for ρ and increases ρ until it succeeds in computing the minimal Z-pair (L, G) for T.

For each value of the guessed order $\rho \in \mathbb{N}$, set

$$F = L \bullet T = (a_\rho \, \partial_x^\rho + \cdots + a_1 \partial_x + a_0) \bullet T$$

where the a_i's are unknown. Since $L \in \mathbb{K}(x)[\partial_x]$, $(L \bullet T)$ is equal to ST with $S \in \mathbb{F}[a_0, \ldots, a_\rho]$ by Lemma 1. Note that the unknowns $a_0, \ldots, a_\rho$ appear in the numerators of S linearly. *Under the assumption that S is nonzero*, F is hyperexponential. Then $\mathcal{Z}$ attempts to compute a Z-pair for T by a variant of Gosper's algorithm. That is, it determines if there exists a hyperexponential G such that $F = \nabla_y \bullet G$. If such a G does not exist, then $\mathcal{Z}$ tries to compute a telescoper of order $\rho + 1$. This process continues until it succeeds in finding a Z-pair for T, provided that such a pair exists.

REMARK 3. For the differential case, $\mathcal{Z}$ is proved to be applicable to all exponential functions [3]. For the shift and the q-shift cases, the fundamental theorem [18, 16] provides a sufficient condition for the termination of $\mathcal{Z}$. It states that if the given (q-)hypergeometric T is proper (see [18, 16] for definitions), then $\mathcal{Z}$ terminates on T. Later, necessary and sufficient conditions for the termination of $\mathcal{Z}$ are presented in [1] for the shift case, and in [6] for the q-shift case.

It is worth noting that the assumption that S is nonzero *may fail* for certain hyperexponential elements. In this case, additional care should be taken, as it will be discussed in the next two sections.

4. A PARTICULAR SET OF HYPEREXPONENTIAL ELEMENTS

In this section we assume $n = 2$, that is, our base field is $\mathbb{F} = \mathbb{C}(x, y)$, and the orthogonal Ore ring $\mathbb{A}_\Delta$ is specialized to $\mathbb{F}[\partial_x, \partial_y; (\sigma_x, \delta_x), (\sigma_y, \delta_y)]$. Furthermore σ_x is either the shift or the q-shift operator if it is not equal to $\mathbf{1}$, and δ_x is the usual differential operator if it is not equal to $\mathbf{0}$. The same assumption applies to σ_y and δ_y.

Let $\mathcal{S}_0$ be the set of all hyperexponential $T(x, y)$'s with the following two properties:

1. T is not hyperexponential-integrable with respect to y.
2. The ∂_x-certificate of T belongs to $\mathbb{C}(x)$.

LEMMA 3. *If $T(x, y) \in \mathcal{S}_0$, then $\mathcal{Z}$ is not applicable to T.*

Proof : Let L of the form (14) be any guessed telescoper for T. By Lemma 1 and the fact that $L \in \mathbb{C}(x)[\partial_x]$, there exists $u \in \mathbb{C}(x)$ such that $L \bullet T = uT$. It follows that

$$\partial_y \bullet (L \bullet T) = \partial_y \bullet (uT) = u(\partial_y \bullet T) = ruT = r(L \bullet T), \quad (16)$$

where r is the ∂_y-certificate of T. Note that r is independent of the guessed telescoper L. Since $T \in \mathcal{S}_0$, (11) at step 0 does not have a rational solution. It follows from (16) that the equations of the form (11) at step 0 and at step ρ are the same. Consequently, $\mathcal{Z}$ is not applicable to T. ∎

Let T belong to $\mathcal{S}_0$ whose ∂_x- and ∂_y-certificates are P and Q, respectively. By Lemma 2

$$P\sigma_x(Q) + \delta_x(Q) = Q\sigma_y(P) + \delta_y(P).$$

Since P is in $\mathbb{C}(x)$, $P\sigma_x(Q) + \delta_x(Q) = PQ$. If $\sigma_x = \mathbf{1}$, then $\delta_x(Q) = 0$. If $\sigma_x \neq \mathbf{1}$, then $\sigma_x(Q) = Q$ because $P \neq 0$ by Definition 5. Consequently, Q is a ∂_x-constant, that is, $Q \in \mathbb{C}(y)$ by the orthogonality of Δ. Intuitively, $\mathcal{S}_0$ consists of hyperexponential elements $f(x)g(y)$, where $f(x)$ and $g(y)$ are hyperexponential with respect to x and y, respectively.

REMARK 4. For the (q-)shift case, it is simple to prove that if the given $T(x, y)$ can be written as $f(x)\, g(y)$, then T is proper. By the fundamental theorem, there exists a Z-pair for T. However, $\mathcal{Z}$ is not applicable to T by Lemma 3 (if T is not (q-)hypergeometric summable with respect to y).

EXAMPLE 2. $\mathcal{Z}$ does not terminate on a number of hypergeometric terms in Gould's *Combinatorial Identities* [9]. This includes the hypergeometric terms used as summands in the formulas *1.105, 1.106, 1.107, 2.20, 2.21, 2.22, 2.23, 3.39, 3.127, 3.128, 10.1, 12.1, 12.2, 24.1, 31.1, 31.2*. All of these hypergeometric terms are elements of $\mathcal{S}_0$.

There are hyperexponential elements outside $\mathcal{S}_0$ to which $\mathcal{Z}$ is not applicable. Since these elements must be annihilated by a nonzero operators in $\mathbb{C}(x)[\partial_x]$, we characterize bivariate hyperexponential elements that are solutions of some operator in $\mathbb{C}(x)[\partial_x]$.

LEMMA 4. *Let $T(x, y)$ be hyperexponential. Then there exists nonzero $L \in \mathbb{C}(x)[\partial_x]$ such that $L \bullet T = 0$ if and only if there exist $\bar{P} \in \mathbb{C}(x)$ and nonzero $a \in \mathbb{C}[x, y]$ such that the ∂_x-certificate P of T can be written as*

$$P = \frac{\sigma_x(a)}{a} \bar{P} + \frac{\delta_x(a)}{a}. \quad (17)$$

Proof. Assume that $L \in \mathbb{C}(x)[\partial_x]$ annihilates T. There are algorithms for computing all hyperexponential solutions of L (see [4] for the differential case, [15] for the shift case, and [2] for the q-shift case). In fact, these algorithms compute the certificate of T with respect to ∂_x. Following the algorithms, we find that the certificate must be of the form $\bar{P} + \delta_x(a)/a$ if $\sigma_x = \mathbf{1}$, and $\bar{P}\sigma_x(a)/a$ if $\delta_x = \mathbf{0}$, where $\bar{P} \in \mathbb{C}(x)$ and a is a polynomial in x whose coefficients possibly involve unspecified constants with respect to (σ_x, δ_x). Since $a \in \mathbb{C}(y)[x]$, we may assume that a belongs to $\mathbb{C}[x, y]$. Hence (17) holds.

Conversely, let P in (17) be the ∂_x-certificate of T, and set $H = T/a$. Note that (5) implies

$$\delta_x\left(\frac{1}{a}\right) = -\frac{\delta_x(a)}{\sigma_x(a)\, a}. \quad (18)$$

We compute

$$\begin{aligned}
\partial_x \bullet H &= \partial_x \left(\tfrac{T}{a}\right) \\
&= \sigma_x\left(\tfrac{1}{a}\right)(\partial_x \bullet T) + \delta_x\left(\tfrac{1}{a}\right) T \quad \text{(by (8))} \\
&= \tfrac{1}{\sigma_x(a)}(\partial_x \bullet T) - \tfrac{\delta_x(a)}{\sigma_x(a) a} T \quad \text{(by (18))} \\
&= \left(\tfrac{1}{\sigma_x(a)} P - \tfrac{\delta_x(a)}{\sigma_x(a) a}\right) aH \quad (\text{since } \partial_x \bullet T = PT) \\
&= \bar{P} H \quad \text{(by (17))}.
\end{aligned}$$

Let $a = \sum_{i=0}^{d} b_i(x) y^i$ with $b_i \in \mathbb{C}[x]$. If $b_j \neq 0$, then

$$L_j = \partial_x - \underbrace{\left(\frac{\sigma_x(b_j)}{b_j}\bar{P} + \frac{\delta_x(b_j)}{b_j}\right)}_{R_j} \in \mathbb{C}(x)[\partial_x]$$

annihilates any hyperexponential element with R_j as its ∂_x-certificate, so it annihilates $y^j b_j H$. Therefore, the least common left multiple (or lclm for short) of the L_j's annihilates aH, which is T. ∎

Let $\mathcal{S}_1$ be the set of all hyperexponential $T(x,y)$'s whose ∂_x-certificates can be written in the form (17). The next lemma describes how elements of $\mathcal{S}_1$ look like.

LEMMA 5. *If $T(x,y) \in \mathcal{S}_1$, then its ∂_y-certificate equals*

$$Q = \frac{\sigma_y(a)}{a}\bar{Q} + \frac{\delta_y(a)}{a} \quad \text{with } \bar{Q} \in \mathbb{C}(y). \tag{19}$$

Consequently, $T \in \mathcal{S}_1$ if and only if

$$T = pH, \tag{20}$$

where $p \in \mathbb{C}[x,y]$ and H is a hyperexponential element whose ∂_x-certificate is in $\mathbb{C}(x)$ and ∂_y-certificate is in $\mathbb{C}(y)$. In addition, the polynomial p may be assumed to be primitive with respect to both x and y.

Proof. Note that Lemma 2 and (9) imply

$$P\sigma_x(Q) + \delta_x(Q) = Q\sigma_y(P) + \delta_y(P). \tag{21}$$

We make a case distinction.

Case 1. If $\sigma_x = \sigma_y = \mathbf{1}$, then (17) and (21) become

$$P = \bar{P} + \frac{\delta_x(a)}{a} \quad \text{and} \quad \delta_x(Q) = \delta_y(P),$$

respectively. It follows that

$$\delta_x(Q) = \delta_y\left(\frac{\delta_x(a)}{a}\right) = \delta_x\left(\frac{\delta_y(a)}{a}\right).$$

Thus $\delta_x\left(Q - \frac{\delta_y(a)}{a}\right) = 0$. The orthogonality of Δ then implies that $\left(Q - \frac{\delta_y(a)}{a}\right)$ is in $\mathbb{C}(y)$, and hence, $Q = \bar{Q} + \frac{\delta_y(a)}{a}$, where $\bar{Q} \in \mathbb{C}(y)$. Hence (19) holds.

Case 2. If $\delta_x = \mathbf{0}$ and $\sigma_y = \mathbf{1}$, then (17) and (21) become

$$P = \frac{\sigma_x(a)}{a}\bar{P} \quad \text{and} \quad P\sigma_x(Q) = QP + \delta_y(P),$$

respectively. It follows that

$$\frac{\sigma_x(a)}{a}\,\sigma_x(Q) = \frac{\sigma_x(a)}{a}Q + \delta_y\left(\frac{\sigma_x(a)}{a}\right),$$

which implies that $\sigma_x\left(Q - \frac{\delta_y(a)}{a}\right) = Q - \frac{\delta_y(a)}{a}$. The orthogonality of Δ then implies that $\left(Q - \frac{\delta_y(a)}{a}\right) \in \mathbb{C}(y)$. Consequently, $Q = \bar{Q} + \frac{\delta_y(a)}{a}$ for some $\bar{Q} \in \mathbb{C}(y)$, and (19) holds.

Similarly, one can show that (19) holds in the cases in which $(\delta_x = \delta_y = \mathbf{0})$ and $(\sigma_x = \mathbf{1}$ and $\delta_y = \mathbf{0})$.

Setting $T = aH_1$ and repeating the computation in the second paragraph of the proof of Lemma 4 for both x and y, one sees that the ∂_x- and ∂_y-certificates of H_1 are in $\mathbb{C}(x)$ and $\mathbb{C}(y)$, respectively. Setting $a = a_1 a_2 p$, where $a_1 \in \mathbb{C}[x]$, $a_2 \in \mathbb{C}[y]$, p is primitive with respect to both x and y, and then $H = a_1 a_2 H_1$, we prove (20). ∎

Note that if $T \in \mathcal{S}_1$, then the minimal annihilator of T in $\mathbb{C}(x)[\partial_x]$ is not necessarily the *minimal* telescoper of T.

EXAMPLE 3. Consider the hypergeometric term

$$T_4(n,k) = (2k^3 - nk^2 - 2n^2k - n^3 + 1)\binom{2k}{k}^3.$$

It is evident that $T_4 \in \mathcal{S}_1$. While the minimal annihilator computed using lclm technique described in the proof of Lemma 4 is of order four, the computed minimal telescoper returned from $\mathcal{Z}$ is of order three.

Let $T(x,y)$ be an element of $(\mathcal{S}_1 \setminus \mathcal{S}_0)$ in form (20). Let $R_0 = 1$, $R_1 \in \mathbb{C}(x)$ be such that $\partial_x \bullet H = R_1 H$. Then one can readily prove that, for $i > 1$, $\partial_x^i \bullet H = R_i H$, where $R_i = \sigma_x(R_{i-1}) R_1 + \delta_x(R_{i-1})$. Note that $R_i \in \mathbb{C}(x)$ by Lemma 5. Consider the application of an operator $L_\rho \in \mathbb{C}(x)[\partial_x]$ of the form (14) to T. We have

$$L_\rho \bullet T = \begin{cases} \underbrace{\left(\sum_{i=0}^{\rho} a_i \sigma_x^i(p) R_i\right)}_{M_\rho} H, & \delta_x = 0, \\ \underbrace{\left(\sum_{i=0}^{\rho} a_i \sum_{j=0}^{i} \binom{i}{j} \left(\delta_x^j(p)\right) R_{i-j}\right)}_{M_\rho} H, & \sigma_x = 1. \end{cases} \tag{22}$$

Observe that $M_\rho \in \mathbb{C}(x)[y]$. Additionally, since the a_i's are different unknowns free of y, $\deg_y M_\rho = \deg_y p \geq 0$. That is, $\deg_y M_\rho$ is independent of ρ. Equation (22) implies

$$\begin{aligned} \partial_y \bullet (L_\rho \bullet T) &= \sigma_y(M_\rho)(\partial_y \bullet H) + \delta_y(M_\rho)\, H \\ &= \underbrace{\left(\frac{\sigma_y(M_\rho)}{M_\rho} R + \frac{\delta_y(M_\rho)}{M_\rho}\right)}_{\bar{R}_\rho}(L_\rho \bullet T) \end{aligned}$$

where R is the ∂_y-certificate of H in $\mathbb{C}(y)$ by Lemma 5. Hence, $\bar{R}_\rho$ is the ∂_y-certificate of $(L_\rho \bullet T)$. Consider the following four cases:

Cases 1 or 2: $(\delta_y = 0$ and $\delta_x = 0)$ or $(\delta_y = 0$ and $\sigma_x = 1)$. Let $(a,b,c) \in \mathbb{C}[y]^3$ be a PNF of R. Then

$$\bar{R}_\rho = \frac{a}{b} \times \frac{\sigma_y(M_\rho c)}{M_\rho c}.$$

Hence, the triple $(a, b, M_\rho c)$ is a PNF of the ∂_y-certificate of $(L_\rho \bullet T)$, and the key equation to be solved (for a polynomial solution $w(y)$) is

$$a(y)\,\sigma_y(w(y)) - \sigma_y^{-1}(b(y))\,w(y) = M_\rho(x,y)\,c(y). \tag{23}$$

Cases 3 or 4: $(\sigma_y = 1$ and $\sigma_x = 1)$ or $(\sigma_y = 1$ and $\delta_x = 0)$.

Let $(a, b, c) \in \mathbb{C}[y]^3$ be a PNF of R. Then

$$\bar{R}_\rho = \frac{a}{b} + \frac{\delta_y(M_\rho c)}{M_\rho c}.$$

Hence, the triple $(a, b, M_\rho c)$ is a PNF of the ∂_y-certificate of $(L_\rho \bullet T)$, and the key equation to be solved (for a polynomial solution $w(y)$) is

$$b(y)\delta_y(w(y)) + (a(y) + \delta_y(b(y)))w(y) = M_\rho(x, y)c(y). \quad (24)$$

From (23), (24), and the fact that $\deg_y M_\rho = \deg_y p$ for all $\rho \in \mathbb{N}$, it follows that the computed upper bounds for the degree of non-zero polynomial solutions of (23) or (24) are the same for all $\rho \in \mathbb{N}$. Let d be the computed upper bound at step $\rho = 0$ in $\mathcal{Z}$. Then there are two cases:

$$d \in \mathbb{N} \quad \text{and} \quad d \notin \mathbb{N}.$$

If $d \in \mathbb{N}$, then set the polynomial solutions of (23) or (24) to be of the form $U_d = u_d y^d + \cdots + u_0$, where $u_d, \ldots, u_0$ are unknown coefficients in $\mathbb{C}(x)$. Substituting U_d into (23) or (24) gives rise to a linear homogeneous system consisting of e equations in $(d+\rho+2)$ unknowns $u_d, \ldots, u_0, a_\rho, \ldots, a_0$, where $e \in \mathbb{Z}^+$ is independent of ρ. The linear system has a solution, in which one of the a_i's, is nonzero when ρ is sufficiently large. Hence, $\mathcal{Z}$ terminates.

If $d \notin \mathbb{N}$, then neither (23) nor (24) has non-zero polynomial solutions for all $\rho \in \mathbb{N}$. $\mathcal{Z}$ does not terminate, because it increases ρ to $(\rho+1)$ as soon as d is not a nonnegative integer for the guessed minimal telescoper L_ρ of the input T. In this case, the minimal telescoper of T has to be the minimal annihilator of T, and we will show how to compute the minimal telescoper efficiently in the next section.

Remark 5. At step $\rho = 0$, the key equations (23) and (24) are of the respective forms

$$a(y)\,\sigma_y(w(y)) - \sigma_y^{-1}(b(y))\,w(y) = p(x, y)\,c(y), \quad (25)$$

and

$$b(y)\,\delta_y(w(y)) + (a(y) + \delta_y(b(y)))\,w(y) = p(x, y)\,c(y). \quad (26)$$

As a consequence, let $\mathcal{S}$ be the union of $\mathcal{S}_0$ and elements T of $(\mathcal{S}_1 \setminus \mathcal{S}_0)$ such that the computed upper bound d for non-zero polynomial solutions of either (25) or (26) is not a nonnegative integer. If $T \in \mathcal{S}$, then $\mathcal{Z}$ does not terminate. On the other hand, if $\mathcal{Z}$ does not terminate on T, then the rational function $R(x, y)$ in (15) equals 0. By Lemma 4, T must be an element of $\mathcal{S}_1$. Since $\mathcal{Z}$ terminates if $T \in \mathcal{S}_1 \setminus \mathcal{S}$, $T \in \mathcal{S}$. This leads to the following main theorem of the paper.

Theorem 1. *Let $T(x, y)$ be a hyperexponential element such that the existence of the minimal Z-pair for T is guaranteed, then $\mathcal{Z}$ does not terminate given T if and only if T is an element of $\mathcal{S}$.*

Example 4. The hypergeometric term T_4 in Example 3 is an element of $\mathcal{S}_1$. Additionally, the computed degree bound for the polynomial $w(y)$ in (25) is 0. Hence, $T_4 \notin \mathcal{S}$, and $\mathcal{Z}$ terminates on T_4. On the other hand, the hypergeometric term T_1 in (2) also belongs to $\mathcal{S}_1$. However, the computed upper bound for the polynomial $w(y)$ in (25) is -1. Hence, $T_1 \in \mathcal{S}$, and $\mathcal{Z}$ does not terminate on T_1.

5. ALGORITHM DESCRIPTION

In this section, we describe an algorithm which helps complete $\mathcal{Z}$. First we show how to determine if a hyperexponential element belongs to $\mathcal{S}_0$ and $\mathcal{S}_1$. Checking whether a hyperexponential T belongs to $\mathcal{S}_0$ is simply based on the definition of $\mathcal{S}_0$. The following lemma provides a formula for computing the minimal Z-pair for every $T \in \mathcal{S}_0$.

Lemma 6. *For $T \in \mathcal{S}_0$, let*

$$\partial_x \bullet T = \frac{s_1}{s_2} T,\ s_1,\ s_2 \in \mathbb{C}[x],\ \gcd(s_1, s_2) = 1. \quad (27)$$

Then $(s_2(x)\,\partial_x - s_1(x),\ 1)$ is the minimal Z-pair for T.

Example 5. The q-hypergeometric term T_2 in (3) is not q-hypergeometric summable with respect to q^k. Additionally, since the ∂_{q^n}-certificate of T_2 equals $q \in \mathbb{C}(q^n)$, $T_2 \in \mathcal{S}_0$. It follows from Lemma 3 that $\mathcal{Z}$ is not applicable to T_2 although $(\partial_{q^n} - q, 1)$ is the minimal Z-pair for T_2 by Lemma 6.

For the recognition of elements in $\mathcal{S}_1$, assume that P is the ∂_x-certificate of T, and the variable y appears in P effectively. Let the denominator of P be fg, where $f \in \mathbb{C}[x]$, $g \in \mathbb{C}[x, y]$, and g is primitive with respect to y.

First, we consider the differential case $\sigma_x = \mathbf{1}$. If T is in $\mathcal{S}_1$, then we may also assume that a in (17) has factors neither in $\mathbb{C}[x]$ nor in $\mathbb{C}[y]$ by Lemma 5. This assumption implies

Lemma 7. *Let $\sigma_x = \mathbf{1}$. If r is a rational function appearing in the irreducible partial fraction decomposition of P with respect to x, then r is either in $\mathbb{C}(x)$ or equal to a logarithmic derivative (with respect to x) of some polynomial with positive degree in y.*

As $f(x)$ and $g(x, y)$, viewed as polynomials in $\mathbb{C}(y)[x]$, are relatively prime, we can decompose

$$P = \frac{f_1}{f} + \frac{g_1}{g} \quad \text{where } f_1, g_1 \in \mathbb{C}(y)[x]. \quad (28)$$

Thus, T does not belong to $\mathcal{S}_1$ if y appears in f_1 effectively by Lemma 7. Assume that f_1 belongs to $\mathbb{C}[x]$. By (17), (28) and the uniqueness of the irreducible partial fraction decomposition, we deduce that T belongs to $\mathcal{S}_1$ if and only if $\frac{g_1}{g} = \frac{\delta_x(a)}{a}$. By Lemma 4, this implies

Lemma 8. *Let $\sigma_x = \mathbf{1}$, and g, g_1, f_1 be given in (28) and f_1 be in $\mathbb{C}[x]$. Then $T \in \mathcal{S}_1$ if and only if the equation*

$$g\,\delta_x(z) = g_1 z \quad (29)$$

has a nonzero polynomial solution in $\mathbb{C}[x, y]$.

Second, we consider the case $\delta_x = \mathbf{0}$. We have

$$P = \frac{f_2}{f}\frac{g_2}{g}, \quad (30)$$

where $f_2 \in \mathbb{C}[x]$ and $g_2 \in \mathbb{C}[x, y]$ primitive w.r.t. y. Once again, if T is in $\mathcal{S}_1$, then we may also assume that a in (17) does not have nontrivial factors in $\mathbb{C}[x]$ or $\mathbb{C}[y]$. From (17), (30) and $\delta_x = \mathbf{0}$, it follows that $\bar{P} = \frac{f_2}{f}$ and $\frac{\sigma_x(a)}{a} = \frac{g_2}{g}$. As a consequence, we have

Lemma 9. *Let $\delta_x = \mathbf{0}$ and g, g_2 be given in (30). Then $T \in \mathcal{S}_1$ if and only if the equation*

$$g\sigma_x(z) = g_2 z \tag{31}$$

has a nonzero polynomial solution in $\mathbb{C}[x, y]$.

There are general algorithms for computing polynomial solutions of linear differential and difference operators. The following ones do the same job for (29) and (31) more efficiently by gcd-calculation.

It is straightforward to derive the following: if (29) has a nonzero polynomial solution a, then (1) $\deg_x g = \deg_x g_1 + 1$; (2) the ratio d of the leading coefficients of g_1 and g with respect to x is a positive integer, which is equal to $\deg_x a$; and (3) d is no less than $\deg_x g$. Assume that these three conditions are satisfied. Then g must divide a over $\mathbb{C}(y)$, because $\gcd(g_1, g) = 1$. By the transformation $z = gu$, we obtain $h\delta_x(u) = h_1 u$, where $h, h_1 \in \mathbb{C}(y)[x]$ with $\gcd(h, h_1) = 1$. The problem is then reduced to finding polynomial solutions of the latter equation with degree no more than $(d - \deg_x g)$.

A similar idea applies to (31). It is easy to see that if (31) has a nonzero polynomial solution a, then (1) $\deg_x g = \deg_x g_2$; (2) the degree of a, say d, can be computed; and (3) d is no less than $\deg_x g$. If $\deg_x g = 0$, we only need to find a polynomial solution of a first-order difference equation with constant coefficients. Otherwise, we can use the same transformation $z = gu$ to reduce the degree bound for the polynomial solution of a new first-order difference equation whose polynomial solutions have a degree bound $(d - \deg_x g)$.

Example 6. For the hypergeometric term T_1 in (2), the polynomials g and g_2 in (30) are $g = (k+1)n^2 + 3k^2 n + k + 1$ and $g_2 = (k+1)n^2 + (3k^2 + 2k + 2)n + 3k^2 + 2k + 2)$. Since $\deg_n g = \deg_n g_2 = 2$, and since the recurrence (31) has a polynomial solution $z = (k+1)n^2 + 3k^2 n + k + 1$, $T_1 \in \mathcal{S}_1$, and can be written in the form (20) where $p = z$ and $H = \binom{2k}{k}^3$. The ∂_k-certificate R of H equals $8(2k+1)^3/(k+1)^3$, which admits $(a, b, c) = (64(k+1/2)^3, (k+1)^3, 1)$ as a PNF with respect to y. Given a, b, c, p, the computed upper bound for $w(y)$ in (25) equals -1. Hence, $\mathcal{Z}$ is not applicable to T_1.

Similarly, for the exponential function T_3 in (4), the polynomials g and g_1 in (28) are $g = x - y - 1$ and $g_1 = 1$. Since the differential equation (29) admits $z = x - y - 1$ as a polynomial solution in $\mathbb{C}[x, y]$, $T_3 \in \mathcal{S}_1$, and can be written in the form (20) where $p = z$ and $H = -y(y+1)\exp\left(\frac{(x-1)^2}{x} + \frac{(y+2)^3}{y}\right)$. The ∂_y-certificate R of H equals $(2y^4 + 8y^3 + 8y^2 - 7y - 8)/(y^2(y+1))$, which admits $(a, b, c) = (2y^3 + 6y^2 + y - 8, y^2, y + 1)$ as a PNF with respect to y. Given a, b, c, p, the computed upper bound for $w(y)$ in (26) equals -1. Hence, $\mathcal{Z}$ is not applicable to T_3.

We now describe a direct algorithm for computing the minimal Z-pairs for elements in $\mathcal{S} \setminus \mathcal{S}_0$. The following theorem provides a sufficient condition for the construction of the minimal Z-pair for a sum of hyperexponential elements based on the minimal Z-pair for each hyperexponential element of the sum. It is a generalization of Lemma 1 and Theorem 2 in [12].

Theorem 2. *Let $(L_1, G_1), \ldots, (L_s, G_s)$ be the minimal* Z*-pairs for the hyperexponential elements $T_1, \ldots, T_s$, respectively. Let $L = \operatorname{lclm}(L_1, \ldots, L_s)$, i.e., there exist $L_1', \ldots, L_s' \in \mathbb{C}(x)[\partial_x]$ such that $L = L_1' L_1 = \cdots = L_s' L_s$. Set $G = L_1' \bullet G_1 + \cdots + L_s' \bullet G_s$. If G is hyperexponential, then (L, G) is a* Z*-pair for $T = T_1 + \cdots + T_s$. Additionally, for any telescoper $\bar{L}$ for T, if $\bar{L}$ is also a telescoper for each T_i, $i = 1, \ldots, s$. Then (L, G) is the minimal* Z*-pair for T.*

Proof. The application of L to T is equal to

$$\sum_{i=1}^{s} L_i' \bullet (L_i \bullet T_i) = \sum_{i=1}^{s} L_i' \bullet (\nabla_y \bullet G_i) = \nabla_y \bullet \left(\sum_{i=1}^{s} L_i' \bullet G_i\right).$$

Hence, $L \bullet T = \nabla_y \bullet G$. The second statement follows from the fact that a telescoper for T is a left multiple of the minimal telescoper for T. ■

For a hyperexponential element T which belongs to $\mathcal{S}$ and which can be written in the form (20), let $p = \sum_{i=0}^{d} b_i(x) y^i$, $b_i \in \mathbb{C}[x]$, and $T_i = b_i\, y^i\, H$ with $b_i \neq 0$.

Lemma 10. *T_i belongs to $\mathcal{S}_0$.*

Proof. Consider the key equations (25) and (26) for T. The corresponding key equations for T_i are the same except for the replacement of the polynomial $p(x, y)$ in the right hand sides by the polynomial $p_i = b_i(x) y^i$. Since $\deg_y p_i \leq \deg_y p$, and since the computed upper bound d for the degree of $w(y)$ in either (25) or (26) is not a non-negative integer, the computed upper bound for the degree of the polynomial $w(y)$ in the corresponding key equations for T_i is not a non-negative integer either. Hence, T_i is not hyperexponential integrable with respect to y. The claim follows since it is evident that the ∂_x-certificate of T_i belongs to $\mathbb{C}(x)$. ■

Lemma 11. *Let $(L_i, 1)$ be the minimal* Z*-pair for T_i. Then $(\operatorname{lclm}(L_i\text{'s}), 1)$ is the minimal* Z*-pair for T.*

Proof. Theorem 2 implies that $L = \operatorname{lclm}(L_i\text{'s})$ is a telescoper for T. Let $\bar{L}$ be any telescoper for T, we need to show that $\bar{L}$ is also a telescoper for each T_i for all i with $0 \leq i \leq d$. Since the ∂_x-certificate of H is free of y by Lemma 5, there exists r_i in $\mathbb{C}(x)$ such that $\bar{L} \bullet T_i = r_i H y^i$. Since T is in $\mathcal{S}$, any telescoper for T is also an annihilator for T. As a consequence, $\bar{L} \bullet T = \sum_{i=1}^{d} (\bar{L} \bullet T_i) = H\left(\sum_{i=1}^{d} r_i y^i\right) = 0$. Thus, r_i is zero, and so is $\bar{L} \bullet T_i$, i.e. $\bar{L}$ is a telescoper for T_i. ■

Lemmas 10 and 11 provide a direct way to compute the minimal Z-pair for an element of $\mathcal{S}$: apply Lemma 6 to compute the minimal telescoper for T_i, $T_i \neq 0$, $0 \leq i \leq d$; compute $L = \operatorname{lclm}(L_i\text{'s})$. Then the minimal Z-pair for T is $(L, 1)$.

EXAMPLE 7. It follows from Example 6 that the hyperexponential T_1 in (2) and T_3 in (4) both belong to $\mathcal{S}$.

The hypergeometric term T_1 can be written in the form (20) where $p = p_1+p_2+p_3$, $H = \binom{2k}{k}^3$, $p_1 = 2nk^2$, $p_2 = (n^2+1)k$, and $p_3 = n^2 + 1$. Applying Lemma 6 results in the minimal telescopers L_i for $p_i H$, $1 \le i \le 3$:

$$L_1 = n\,\partial_n - (n+1), \quad L_2 = L_3 = (n^2+1)\,\partial_n - (n^2+2n+2).$$

Hence, $(\mathrm{lclm}\,(L_1, L_2), 1)$ is the minimal Z-pair for T_1 where

$$\mathrm{lclm}\,(L_1, L_2) = (n^2+n-1)\partial_n^2 - (2n^2+4n-2)\partial_n + (n^2+3n+1).$$

The exponential function T_3 can be written in the form (20) where $p = p_1 + p_2$, $H = -y(y+1)\exp\left(\frac{(x-1)^2}{x} + \frac{(y+2)^3}{y}\right)$, $p_1 = -y$, and $p_2 = x - 1$. Applying Lemma 6 results in the minimal telescopers L_i for $p_i H$, $1 \le i \le 2$:

$$L_1 = x^2\,\partial_x - (x-1)(x+1), \quad L_2 = (x-1)x^2\,\partial_x - x^3 + x - 1.$$

Hence, $(\mathrm{lclm}\,(L_1, L_2), 1)$ is the minimal Z-pair for T_3 where

$$\mathrm{lclm}\,(L_1, L_2) = x^4\partial_x^2 - 2x^2(x-1)(x+1)\partial_x + x^4 - 2x^2 - 2x + 1.$$

We conclude this paper with a description of a modification to $\mathcal{Z}$ which guarantees to compute the minimal Z-pair for a given hyperexponential element, provided that such a pair exists. This modification also guarantees that the two statements (A) and (B) given in Section 1 become equivalent.

Apply Lemma 8 (differential) or Lemma 9 ((q-)shift) to determine if a given hyperexponential T belongs to $\mathcal{S}_1$.

1. If $T \in \mathcal{S}_1$, rewrite T in the form (20) where p equals z in (29) (differential) or in (31) ((q-)shift), and compute the upper bound d for the degree of the polynomial $w(y)$ in the key equation (25) (differential) or (26) ((q-)shift). If d is not a non-negative integer, then $T \in \mathcal{S}$, and the minimal Z-pair for T can be directly computed using the lclm technique (Lemma 11); otherwise, use $\mathcal{Z}$ to compute the minimal Z-pair for T.

2. If $T \notin \mathcal{S}_1$, use $\mathcal{Z}$ to compute the minimal Z-pair for T.

The modified version of $\mathcal{Z}$ described above is implemented for the differential case and for the shift case in the computer algebra system Maple, and is available from `http://www.cecm.sfu.ca/~hle/maple/Zcomplete/`.

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