The PolynomialIdeals Maple Package

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Abstract

PolynomialIdeals is a new Maple package for computing with ideals in a polynomial ring. We introduce the package, describe some of its innovative features, and show how it can be used to solve systems of polynomial equations. We also present the algorithms for primary decomposition and radical computation, since our implementations differ from what has been advanced in the literature.

Introduction

The PolynomialIdeals package provides a suite of efficient algorithms for computing with ideals of polynomial rings over a field. Much of this functionality is new to Maple, being previously available only to users of specialized computer algebra systems, such as Magma and Singular. A preliminary version of the package is installed in Maple 9.5, but development is ongoing. The latest version can be downloaded from
`http://www.cecm.sfu.ca/~rpearcea`.

Using the Package

Definition. Let R be a commutative ring, a subset $I \subset R$ is an *ideal* if

1. $f + g \in I$ for all $f, g \in I$
2. $f\,h \in I$ for all $f \in I$ and $h \in R$

In our case $R = k[x_1, \ldots, x_n]$, the polynomial ring in n variables over a field k. We begin by loading the package using the `with` command. To create an ideal, simply enclose its generators within angled brackets.

```
> with(PolynomialIdeals);
```

```
Warning, the assigned names <,>, NormalForm and UnivariatePolynomial
now have a global binding
```

[<, >, *Add*, *Contract*, *EliminationIdeal*, *Generators*, *GroebnerBasis*, *HilbertDimension*, *IdealContainment*, *IdealInfo*, *IdealMembership*, *Intersect*, *IsMaximal*, *IsPrimary*, *IsPrime*, *IsProper*, *IsRadical*, *LeadingMonomial*, *MaximalIndependentSet*, *Multiply*, *NormalForm*, *Operators*, *PolynomialIdeal*, *PrimaryDecomposition*, *PrimeDecomposition*, *Quotient*, *Radical*, *Saturate*, *SimplifyIdeal*, *SuggestVariableOrder*, *UnivariatePolynomial*, *ZeroDimensionalDecomposition*, *in*]

```
> J := <x^2*z-y^2, x^2-y^2*z>;
```

$$J := \langle x^2\, z - y^2,\, x^2 - y^2\, z \rangle$$

The coefficients are assumed to be rational numbers by default, and any indeterminates which appear are considered variables of the polynomial ring, so this ideal resides in $\mathbb{Q}[x, y, z]$. We can construct ideals in other rings using the optional arguments `(characteristic=p)` and `(variables={...})`. The next example constructs an ideal in $\mathbb{Z}_5(a)[x, y]$. The `IdealMembership` command tests polynomials for membership in an ideal.

```
> K := <x^2-a*y+a^3,a*x*y-1,(characteristic=5,variables={x,y})>;
```

$$K := \langle x^2 + 4ay + a^3, axy + 4 \rangle$$

```
> IdealMembership(a^2*y^3-a^4*y^2-x*y, K);
```

$$true$$

In the computation above, a Gröbner basis was computed for K automatically. Gröbner bases are stored directly in the ideal data structure and reused whenever possible during subsequent computations. On large systems this produces significant savings in time. In the example below we decompose the katsura-4 system into an intersection of four prime components. The radical test which follows is instantaneous.

```
> katsura4 := <2*t+u+2*x+2*y+2*z-1, 2*x*t+2*u*y+2*z*t-y,
> 2*t^2+u^2+2*x^2+2*y^2+2*z^2-u, 2*t*u+x*y+2*z*t+2*y*z-t,
> t^2+2*y*t+2*z*u+2*x*z-z>:
```

```
> SimplifyIdeal(PrimaryDecomposition(katsura4));
```

$$
\begin{aligned}
&\langle x,\, t,\, y,\, z,\, u-1\rangle,\ \langle 3\,u-1,\, t,\, y,\, z,\, 3\,x-1\rangle,\\
&\langle 28\,z^2-12\,z+1,\, t,\, u-2\,z,\, y,\, 2\,x+4\,z-1\rangle,\\
&\langle 2\,u-1,\, 4\,t+1,\, 2\,y-1,\, z+x,\, 32\,z^2+3\rangle, \langle 40692092\,t^2\,u-105009548\,t^3\\
&-68613814\,t\,u-10030652\,y\,t-53644024\,z\,t-38567838\,t^2+2681921\,u\\
&-716492\,y-654138\,z+47103408\,t-1259797,\\
&280\,y^2-102\,t\,u+660\,y\,t-252\,z\,t+442\,t^2-11\,u-98\,y+2\,z-72\,t+7,\\
&10\,z\,u+39\,t\,u+10\,y\,t+24\,z\,t+16\,t^2-3\,u+y-4\,z-21\,t+1,\\
&13044680499934600\,t^4-8556027057302680\,t^3-3578757194560758\,t\,u\\
&-501760691393260\,y\,t-3254487989979588\,z\,t-2095088940432442\,t^2\\
&+151144400960241\,u-53209818728462\,y-38482104708202\,z\\
&+2729106609887472\,t-73450460152097,\\
&7\,u^2-52\,t\,u-16\,y\,t-28\,z\,t-22\,t^2-3\,u-2\,z+30\,t, 40692092\,z\,t^2\\
&+165357276\,t^3+91414257\,t\,u+15750880\,y\,t+76052144\,z\,t+25036598\,t^2\\
&-3710605\,u+1015455\,y+909676\,z-64918943\,t+1760115, 406920920\,y\,t^2\\
&+1514910180\,t^3+644633608\,t\,u+39424940\,y\,t+600529328\,z\,t+51085042\,t^2\\
&-25722351\,u+8516802\,y+5023342\,z-447608702\,t+12129467,\\
&2\,t+u+2\,x+2\,y+2\,z-1,\\
&280\,y\,z+388\,t\,u+240\,y\,t+308\,z\,t+302\,t^2-11\,u-28\,y+2\,z-282\,t+7,\\
&2\,u\,y-t\,u-2\,y\,t-2\,t^2-y+t,\\
&-31\,u-12\,t+42\,y-58\,z-218\,t^2+280\,z^2+308\,z\,t+158\,t\,u-380\,y\,t+7\rangle
\end{aligned}
$$

```
> IsRadical(katsura4);
```

$$\mathit{true}$$

```
> HilbertDimension(katsura4);
```

$$0$$

Gröbner bases can also be computed explicitly. The syntax for monomial orders is a superset of Maple's `Groebner` package. Below we compute a Gröbner basis for an elimination order with $\{x,u\} >> \{y,z\} >> t$. Instead of using Buchberger's algorithm, the Gröbner basis is converted automatically using the FGLM algorithm [3]. In this example the speedup is about a factor of fifty. Future versions of this package will contain the Gröbner Walk algorithm [2], so that non zero-dimensional systems can be converted as well.

Next we compute the intersection of the ideal with the subring $\mathbb{Q}[y, z, t]$. This is instantaneous because a suitable Gröbner basis is already known.

```
>  GroebnerBasis(katsura4, lexdeg([x,u],[y,z],[t])):
>  EliminationIdeal(katsura4,{y,z,t}):
```

By default, prime and primary decompositions are computed over the field implied by the coefficients of the generators. Algebraic field extensions can also be specified directly, similiar to Maple's `factor` command.

```
>  J := <x^2+6>;
```

$$J := \langle x^2 + 6 \rangle$$

```
>  SimplifyIdeal(PrimeDecomposition(J, {I,sqrt(2),RootOf(z^2-3)}));
```

$$\langle x + \sqrt{2}\,\mathrm{RootOf}(\_Z^2 - 3)\, I \rangle,\ \langle x - \sqrt{2}\,\mathrm{RootOf}(\_Z^2 - 3)\, I \rangle$$

```
>  K := <x^3-5, y^2+3, (characteristic=13)>;
```

$$K := \langle x^3 + 8,\ y^2 + 3 \rangle$$

```
>  SimplifyIdeal(PrimaryDecomposition(K, sqrt(5)));
```

$$\langle y + 6,\ x + 6 \rangle,\ \langle x + 6,\ y + 7 \rangle,\ \langle y + 6,\ x + 2 \rangle,\ \langle x + 5,\ y + 7 \rangle,\ \langle y + 7,\ x + 2 \rangle,\ \langle x + 5,\ y + 6 \rangle$$

Solving Polynomial Systems

The PolynomialIdeals package contains a number of routines that will be of interest to anyone solving large polynomial systems in Maple. In this section we demonstrate two techniques for pre-processing systems which are particularly useful.

The most basic technique is to compute a lexicographic Gröbner basis prior to calling `solve`. For zero-dimensional systems this computation uses FGLM, which is typically much faster than the Buchberger algorithm. The system below is otherwise intractable in Maple 9.

```
>  cassou;
```

$$\{261 + 4\,d\,b^2\,c - 3\,d^2\,b^2 - 4\,c^2\,b^2 + 22\,e\,c - 22\,d\,e, 216\,d\,b^2\,c - 162\,d^2\,b^2 - 81\,c^2\,b^2 + 5184 + 1008\,e\,c - 1008\,d\,e + 15\,c^2\,b^2\,d\,e - 15\,c^3\,b^2\,e - 80\,d\,e^2\,c + 40\,d^2\,e^2 + 40\,e^2\,c^2, 15\,b^4\,c\,d^2 + 6\,b^4\,c^3 + 21\,b^4\,c^2\,d - 144\,b^2\,c - 8\,b^2\,c^2\,e - 28\,b^2\,c\,d\,e - 648\,b^2\,d + 36\,b^2\,d^2\,e + 9\,b^4\,d^3 - 120, 30\,c^3\,b^4\,d - 32\,d\,e^2\,c - 720\,d\,b^2\,c - 24\,c^3\,b^2\,e - 432\,c^2\,b^2 + 576\,e\,c - 576\,d\,e + 16\,c\,b^2\,d^2\,e + 16\,d^2\,e^2 + 16\,e^2\,c^2 + 9\,c^4\,b^4 + 5184 + 39\,d^2\,b^4\,c^2 + 18\,d^3\,b^4\,c - 432\,d^2\,b^2 + 24\,d^3\,b^2\,e - 16\,c^2\,b^2\,d\,e - 240\,c\}$$

```
> TIMER:=time():
> cassou := <cassou>:
> SuggestVariableOrder(cassou);
```

$$e,\ d,\ c,\ b$$

```
> G := GroebnerBasis(cassou, plex(e,d,c,b)):
> solve({op(G)}):
> time()-TIMER;
```

$$2.810$$

The second technique is to remove solutions which are not of interest in hopes of simplifiying the problem. The system below is from [6]. It describes an optimal packing of 10 identical circles in a square where the variable m is proportional to the radius. The goal is to compute a univariate polynomial in m, which can then be solved to produce the maximum possible radius. The difficulty lies in the fact that the system is not zero-dimensional, so that the FGLM algorithm can not be used.

```
> circles;
```

$$\langle x^2 + y^2 - m^2,\ 2 - 2\,z + z^2 + w^2 + 2\,w\,m - 2\,w - 2\,m,\ 1 - 2\,x - 2\,m + x^2 + 2\,x\,m + w^2, z^2 + 1 - 2\,y - 2\,m + y^2 + 2\,y\,m - 3\,m^2,\ 5 - 6\,m - 4\,z + m^2 + 4\,m\,z + z^2 + 4\,w^2 - 4\,w - 4\,w\,y - 4\,w\,m + 2\,y + y^2 + 2\,y\,m\rangle$$

```
> HilbertDimension(circles);
```

$$1$$

```
> newcircles := SimplifyIdeal(circles, m<>0):
> HilbertDimension(newcircles);
```

$$0$$

```
> UnivariatePolynomial(m, newcircles);
```

$$98015844\,m^{16} + 1526909568\,m^{11} + 114038784\,m^{2} + 1145811528\,m^{14} - 563649536\,m^{3}$$
$$- 1038261808\,m^{10} - 11436428\,m^{17} - 9722063488\,m^{7} - 4564076288\,m^{5}$$
$$+ 7918461504\,m^{6} + 227573920\,m^{12} - 14172160\,m - 2960321792\,m^{9}$$
$$+ 7803109440\,m^{8} + 819200 - 1398966480\,m^{13} + 1180129\,m^{18} - 462103584\,m^{15}$$
$$+ 1899131648\,m^{4}$$

Radicals and Primary Decomposition

For many basic operations the PolynomialIdeals package relies on the standard algorithms given in [1]. For primary decomposition however, this approach was wholly inadequate and an alternative had to be found. The following result of [4] forms the basis of our current algorithms.

Theorem. *Let $I \subset k[x_1, \ldots, x_n]$ be a zero-dimensional ideal, and let G be a reduced Gröbner basis for I under lexicographic order with $x_1 > x_2 > \cdots > x_n$. Then for each i, G contains a unique polynomial $g_i \in I \cap k[x_i, \ldots, x_n]$ whose leading monomial is a power of x_i. If, for all i, g_i is irreducible mod $\sqrt{I \cap k[x_{i-1}, \ldots, x_n]}$ then I is a prime ideal.*

The implications of this theorem are obvious, because there are only two reasons why a polynomial might fail to be irreducible. First, g_i could be a power of an irreducible, in which case the ideal is still primary and we can continue with the next variable. Second, the polynomial could factor as $g_i = \prod_{i=1}^{t} f_i^{e_i}$ in which case the ideal factors as $I = \bigcap_{i=1}^{t} (I + \langle f_i^{e_i} \rangle)$. To compute a prime decomposition, we can simply remove the exponents.

Combined with the FGLM algorithm and Maple's excellent `factor` command, this method of primary decomposition is very effective. An even bigger improvement can be made to radical computations, however. The traditional algorithm for computing the radical of a zero-dimensional ideal dates back to Seidenberg [5]. We compute univariate polynomials in each variable, typically using FGLM, and add their square-free parts to the generating set. The problem with this method is that in practice, computation of a single univariate polynomial is a majority of the work towards computing a full lexicographic Gröbner basis. Furthermore, for some variables the univariate polynomial computation may produce disasterous coefficient growth.

A better method is thus desired for systems with high degree and a large number of variables. We observe the following. If we were to proceed instead with a prime decomposition using the method above, we would encounter a polynomial

$$g_i = \prod_{i=1}^{t} f_i^{e_i} \mod \sqrt{I \cap k[x_{i-1}, \ldots, x_n]}$$

At this point

$$\begin{aligned} \sqrt{I} \cap k[x_{i-1}, \ldots, x_n] &= \bigcap_{i=1}^{t} (I + \langle f_i \rangle) \cap k[x_{i-1}, \ldots, x_n] \\ &= (I + \langle \prod_{i=1}^{t} f_i \rangle) \cap k[x_{i-1}, \ldots, x_n] \end{aligned}$$

so we could strip away the exponents and compute the radical of I. However, this doesn't require a complete factorization at all – we need only add the square-free part of each g_i to make the ideal radical. The resulting algorithm performs one total degree Gröbner basis computation one conversion to a lexicographic order of our choice. The time required to compute the square-free parts is insignificant.

Benchmarks

Everybody loves benchmarks, especially when they contribute to an ongoing war over which computer algebra system is better. Setting aside issues of fairness and the relative speed of compiled versus interpreted code, we will compare the current version of PolynomialIdeals running on Maple 9 to version 2.10-8 of the Magma computer algebra system. The tests were performed on a 2.8 GHz Pentium 4 Xeon machine with 4 GB of memory. All times are in seconds.

In the tests below, we have tried our best to be a gracious competitor. PolynomialIdeals chooses heuristically optimal variable orders in all its computations, so we have been careful to use the same orders in Magma. This test focuses entirely on zero-dimensional systems, since our implementation of the Gröbner Walk is still preliminary.

Primary Decomposition							
	cyclic6	eco7	rose	katsura6	reimer5	virasoro	rbpl24
Maple	74.87	85.55	52.83	188	717	1607	2244
Magma	.360	.120	.950	25.25	103	23.1	561.3

Is Radical							
	cyclic6	eco7	rose	katsura6	reimer5	virasoro	rbpl24
Maple	65.16	83.3	48.08	165.5	452.5	1523	2232
Magma	1.17	2.11	47.31	250.1	9.8	168.64	6144

The times for cyclic6 and eco7 clearly demonstrate the advantage Magma enjoys due to its compiled Gröbner basis implementation. The rose system illustrates our point about the relative difficulty of computing all the univariate polynomials. In this case there are only three. The katsura6 system is a generically nasty example. Not impossible, but its high degree and bad coefficient growth are enough to tax any computer algebra system. PolynomialIdeals makes a good showing, and our radical algorithm proves its point once again. Reimer5 is another difficult system, but one in which the univariate polynomials have significantly lower degree than the rest of the system. Needless to say, the standard radical algorithm demolishes our method in this particular case.

Virasoro is a difficult system for PolynomialIdeals, which gets bogged down by horrendous coefficient blowup in both the Buchberger algorithm and FGLM. Magma's primary decomposition does extremely well. The last system is rbpl24, a nine variable system from robotics included to test asymptotic behavior. PolynomialIdeals finishes within a factor of four, and our radical algorithm demonstrates its worth conclusively.

References

[1] T. Becker, V. Weispfenning. *Gröbner Bases: A Computational Approach to Commutative Algebra.* Springer-Verlag, New York Berlin Heidelberg, 1993

[2] S. Collart, M. Kalkbrener, D. Mall. *Converting Bases with the Gröbner Walk.* Journal of Symbolic Computation, 24, pp. 465-469, 1997

[3] J.C. Faugère, P. Gianni, D. Lazard, T. Mora. *Efficient Computation of Zero Dimensional Gröbner Bases by Change of Ordering*. Journal of Symbolic Computation, 16, pp. 329-344, 1994

[4] P. Gianni, B. Trager, G. Zacharias. *Grobner bases and Primary Decomposition of Polynomial Ideals*. Journal of Symbolic Computation, 6, pp. 149-167, 1988

[5] A. Seidenberg. *Constructions in Algebra*. Trans. Amer. Math. Soc., 197, pp. 272-313, 1974

[6] D. Würtz, M. Monagan, R. Peikert *The History of Packing Circles in a Square* Maple in Mathematics and the Sciences – A Special Issue of the Maple Technical Newsletter, pp. 35-42, 1994

Appendix

Here we collect the various systems that were used in the benchmarks. All of them are available in electronic form from either
`http://www2.math.uic.edu/~jan/Demo/TITLES.html` or
`http://www.symbolicdata.org/SD_HTML/index.html`.
The second link is not a direct one, follow the links for "Data" and "INTPS". We would like to thank the authors of these pages for providing a valuable resource.

```
> cyclic6;
```

$$\langle x1\,x2\,x3\,x4\,x5\,x6-1,\ x1+x2+x3+x4+x5+x6,\\ x1\,x2+x2\,x3+x3\,x4+x4\,x5+x1\,x6+x5\,x6,\\ x1\,x2\,x3+x2\,x3\,x4+x3\,x4\,x5+x1\,x2\,x6+x1\,x5\,x6+x4\,x5\,x6,\\ x1\,x2\,x3\,x4+x2\,x3\,x4\,x5+x1\,x2\,x3\,x6+x1\,x2\,x5\,x6+x1\,x4\,x5\,x6+x3\,x4\,x5\,x6,\\ x1\,x2\,x3\,x4\,x5+x1\,x2\,x3\,x4\,x6+x1\,x2\,x3\,x5\,x6+x1\,x2\,x4\,x5\,x6+x1\,x3\,x4\,x5\,x6\\ +x2\,x3\,x4\,x5\,x6\rangle$$

```
> eco7;
```

$$\langle x7\,x4+x7\,x1\,x5+x7\,x6\,x2-4,\ x7\,x5+x1\,x6\,x7-5,\ x6\,x7-6,\\ x1+x2+x3+x4+x5+x6+1,\\ x1\,x7+x1\,x2\,x7+x7\,x2\,x3+x7\,x3\,x4+x7\,x4\,x5+x5\,x6\,x7-1,\\ x7\,x2+x7\,x1\,x3+x7\,x2\,x4+x7\,x3\,x5+x7\,x6\,x4-2,\\ x7\,x3+x7\,x1\,x4+x7\,x2\,x5+x7\,x6\,x3-3\rangle$$

```
> rose;
```

$$\langle 7\,y^4-20\,x^2,\ 2160\,x^2\,z^4+1512\,x\,z^4+315\,z^4-4000\,x^2-2800\,x-490, 40320000\,x^6\,y^2\,z\\ +67200000\,x^5\,y^3+28800000\,x^5\,y^2\,z+94080000\,x^4\,y^3-23520000\,x^4\,y\,z^2\\ -10080000\,x^4\,z^3+40924800\,x^3\,y^3+21168000\,x^3\,y^2\,z-41395200\,x^3\,y\,z^2\\ -28224000\,x^3\,z^3+2634240\,x^2\,y^3+4939200\,x^2\,y^2\,z-26726560\,x^2\,y\,z^2\\ -15288000\,x^2\,z^3-2300844\,x\,y^3+347508\,x\,y^2\,z-7727104\,x\,y\,z^2-1978032\,x\,z^3\\ -432180\,y^3-852355\,y\,z^2-180075\,z^3\rangle$$

```
> katsura6;
```

$$\langle x0+2\,x1+2\,x2+2\,x3+2\,x4+2\,x5+2\,x6-1,\\ 2\,x2\,x3+2\,x1\,x4+2\,x0\,x5+2\,x1\,x6-x5,\\ x2^2+2\,x1\,x3+2\,x0\,x4+2\,x1\,x5+2\,x6\,x2-x4,\\ 2\,x1\,x2+2\,x0\,x3+2\,x1\,x4+2\,x2\,x5+2\,x6\,x3-x3,\\ x1^2+2\,x0\,x2+2\,x1\,x3+2\,x2\,x4+2\,x3\,x5+2\,x6\,x4-x2,\\ 2\,x0\,x1+2\,x1\,x2+2\,x2\,x3+2\,x3\,x4+2\,x4\,x5+2\,x5\,x6-x1,\\ x0^2+2\,x1^2+2\,x2^2+2\,x3^2+2\,x4^2+2\,x5^2+2\,x6^2-x0\rangle$$

```
> reimer5;
```

$$\langle -1+2\,x^2-2\,y^2+2\,z^2-2\,t^2+2\,u^2,\ -1+2\,x^3-2\,y^3+2\,z^3-2\,t^3+2\,u^3,$$
$$-1+2\,x^4-2\,y^4+2\,z^4-2\,t^4+2\,u^4,\ -1+2\,x^5-2\,y^5+2\,z^5-2\,t^5+2\,u^5,$$
$$-1+2\,x^6-2\,y^6+2\,z^6-2\,t^6+2\,u^6\rangle$$

```
> virasoro;
```

$$\langle -6\,x4\;x5+6\,x4\;x8+6\,x5\;x8-6\,x6\;x7+6\,x6\;x8+6\,x7\;x8+8\,x8^2-x8, 8\,x1^2+8\,x1\;x2$$
$$+8\,x1\;x3+2\,x1\;x4+2\,x1\;x5+2\,x1\;x6+2\,x1\;x7-8\,x2\;x3-2\,x7\;x4-2\,x5\;x6$$
$$-x1, 8\,x1\;x2-8\,x1\;x3+8\,x2^2+8\,x2\;x3+2\,x2\;x4+2\,x2\;x5+2\,x6\;x2+2\,x7\;x2$$
$$-2\,x6\;x4-2\,x7\;x5-x2, -8\,x1\;x2+8\,x1\;x3+8\,x2\;x3+8\,x3^2+2\,x3\;x4+2\,x3\;x5$$
$$+2\,x6\;x3+2\,x7\;x3-2\,x4\;x5-2\,x6\;x7-x3, 2\,x1\;x4-2\,x1\;x7+2\,x2\;x4-2\,x6\;x2$$
$$+2\,x3\;x4-2\,x3\;x5+8\,x4^2+8\,x4\;x5+2\,x6\;x4+2\,x7\;x4+6\,x4\;x8-6\,x5\;x8-x4,$$
$$2\,x1\;x5-2\,x1\;x6+2\,x2\;x5-2\,x7\;x2-2\,x3\;x4+2\,x3\;x5+8\,x4\;x5-6\,x4\;x8$$
$$+8\,x5^2+2\,x5\;x6+2\,x7\;x5+6\,x5\;x8-x5, -2\,x1\;x5+2\,x1\;x6-2\,x2\;x4+2\,x6\;x2$$
$$+2\,x6\;x3-2\,x7\;x3+2\,x6\;x4+2\,x5\;x6+8\,x6^2+8\,x6\;x7+6\,x6\;x8-6\,x7\;x8-x6,$$
$$-2\,x1\;x4+2\,x1\;x7-2\,x2\;x5+2\,x7\;x2-2\,x6\;x3+2\,x7\;x3+2\,x7\;x4+2\,x7\;x5$$
$$+8\,x6\;x7-6\,x6\;x8+8\,x7^2+6\,x7\;x8-x7\rangle$$

```
> rbpl24;
```

$$
\begin{aligned}
&\langle 62500\,x1^{2} + 62500\,y1^{2} + 62500\,z1^{2} - 74529,\ 3200\,x2 + 1271,\\
&625\,x2^{2} + 625\,y2^{2} + 625\,z2^{2} - 1250\,x2 - 2624,\\
&12500\,x3^{2} + 12500\,y3^{2} + 12500\,z3^{2} + 2500\,x3 - 44975\,y3 - 10982,\\
&400000\,x1\ x2 + 400000\,y1\ y2 + 400000\,z1\ z2 - 400000\,x2 + 178837,\\
&1000000\,x1\ x3 + 1000000\,y1\ y3 + 1000000\,z1\ z3 + 100000\,x3 - 1799000\,y3\\
&- 805427, 2000000\,x2\ x3 + 2000000\,y2\ y3 + 2000000\,z2\ z3 - 2000000\,x2\\
&+ 200000\,x3 - 3598000\,y3 - 1403, 113800000000000\,x2\ y1\ z3\\
&- 113800000000000\,x2\ y3\ z1 - 113800000000000\,x3\ y1\ z2\\
&+ 113800000000000\,x1\ y3\ z2 - 113800000000000\,x1\ y2\ z3 + 1475978220000\,y2\\
&- 151600474800000\,z2 + 292548849600000\,x3 - 825269402280000\,y3\\
&+ 825859951200000\,z3 - 206888400000000\,y2\ x3 - 362960716800000\,x1\\
&+ 38025201600000\,x2 + 113800000000000\,x3\ y2\ z1 - 1212982689600000\,z1\\
&+ 11809567440000\,y1 - 206888400000000\,x2\ y1 + 206888400000000\,x3\ y1\\
&+ 206888400000000\,x1\ y2 - 206888400000000\,x1\ y3 + 206888400000000\,x2\ y3\\
&- 2014260000000\,x2\ z1 + 2014260000000\,x3\ z1 - 61907200000000\,y2\ z1\\
&+ 61907200000000\,y3\ z1 + 2014260000000\,x1\ z2 - 2014260000000\,x3\ z2\\
&+ 61907200000000\,y1\ z2 - 61907200000000\,y3\ z2 - 2014260000000\,x1\ z3\\
&+ 2014260000000\,x2\ z3 - 61907200000000\,y1\ z3 + 61907200000000\,y2\ z3\\
&- 19295432410527, -777600000000\,x2\ y1\ z3 + 777600000000\,x2\ y3\ z1\\
&+ 777600000000\,x3\ y1\ z2 - 777600000000\,x1\ y3\ z2 + 777600000000\,x1\ y2\ z3\\
&- 268090368000\,y2 + 354583756800\,z2 + 158626915200\,x3 + 72704002800\,y3\\
&+ 307085438400\,z3 - 1409011200000\,y2\ x3 + 235685027200\,x1\\
&+ 398417510400\,x2 - 777600000000\,x3\ y2\ z1 + 412221302400\,z1\\
&- 311668424000\,y1 + 282499646407 - 1409011200000\,x2\ y1\\
&+ 1409011200000\,x3\ y1 + 1409011200000\,x1\ y2 - 1409011200000\,x1\ y3\\
&+ 1409011200000\,x2\ y3 - 1065312000000\,x2\ z1 + 1065312000000\,x3\ z1\\
&- 805593600000\,y2\ z1 + 805593600000\,y3\ z1 + 1065312000000\,x1\ z2\\
&- 1065312000000\,x3\ z2 + 805593600000\,y1\ z2 - 805593600000\,y3\ z2\\
&- 1065312000000\,x1\ z3 + 1065312000000\,x2\ z3 - 805593600000\,y1\ z3\\
&+ 805593600000\,y2\ z3\rangle
\end{aligned}
$$