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Implementing Kaltofen and Yagati's fast transposed Vandermonde solver

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Outline

1. Transposed Vandermonde System
2. Fast Division
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Transposed Vandermonde System of Equations

Let F be a field. Let $a = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1} \in F[x]$ where all a_is are unknown. Given $u_1, u_2, \ldots, u_n \in F$, let $b_i = a(u_i)$ for $1 \leq i \leq n$.

Let $u_i = \alpha^{i-1}$ for some $\alpha \in F$ where $\alpha^i \neq \alpha^j$ for all $i \neq j$. Then

$$b_i = a(u_i) = \sum_{j=0}^{n-1} a_j (\alpha^{i-1})^j = \sum_{j=0}^{n-1} a_j (\alpha^j)^{i-1} = \sum_{j=0}^{n-1} a_j (u_{j+1})^{i-1} \text{ for } 1 \leq i \leq n.$$

Problem: Interpolate a with $u_1, u_2, \ldots, u_n$ and $b_1, b_2, \ldots, b_n$.

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ u_1 & u_2 & u_3 & \cdots & u_n \\ u_1^2 & u_2^2 & u_3^2 & \cdots & u_n^2 \\ \vdots & \vdots & \vdots & & \vdots \\ u_1^{n-1} & u_2^{n-1} & u_3^{n-1} & \cdots & u_n^{n-1} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix}}_{\mathbf{b}}$$

Transposed Vandermonde System of Equations (continued)

We call $U\mathbf{a} = \mathbf{b}$ a transposed Vandermonde system of equations.

Methods	# ops in F	space
Gaussian Elimination	$O(n^3)$	$O(n^2)$
Zippel's method [6]	$O(n^2)$	$O(n)$
Kaltofen & Yagati's method [4]	$O(M(n)\log n)$	$O(n\log n)$

$M(n)$ is the number of field operations for polynomial multiplication with two polynomials at most n in $F[x]$.

Goal: an algorithm to solve transposed Vandermonde system which does $O(n\log^2 n)$ field operations in F.

Fast multiplication

The in-place recursive FFT from Law and Monagan [5] which does exactly $\frac{1}{2} n \log_2 n$ multiplications.

- An array W of size n of the powers of ω needed for all recursive calls
- The two bit-reversal permutations cancellation.

The three primes method: Let $f, g \in \mathbb{Z}_p[x]$. To compute $f \times g \in \mathbb{Z}_p[x]$

- Choose three Fourier primes p_1, p_2, p_3 such that

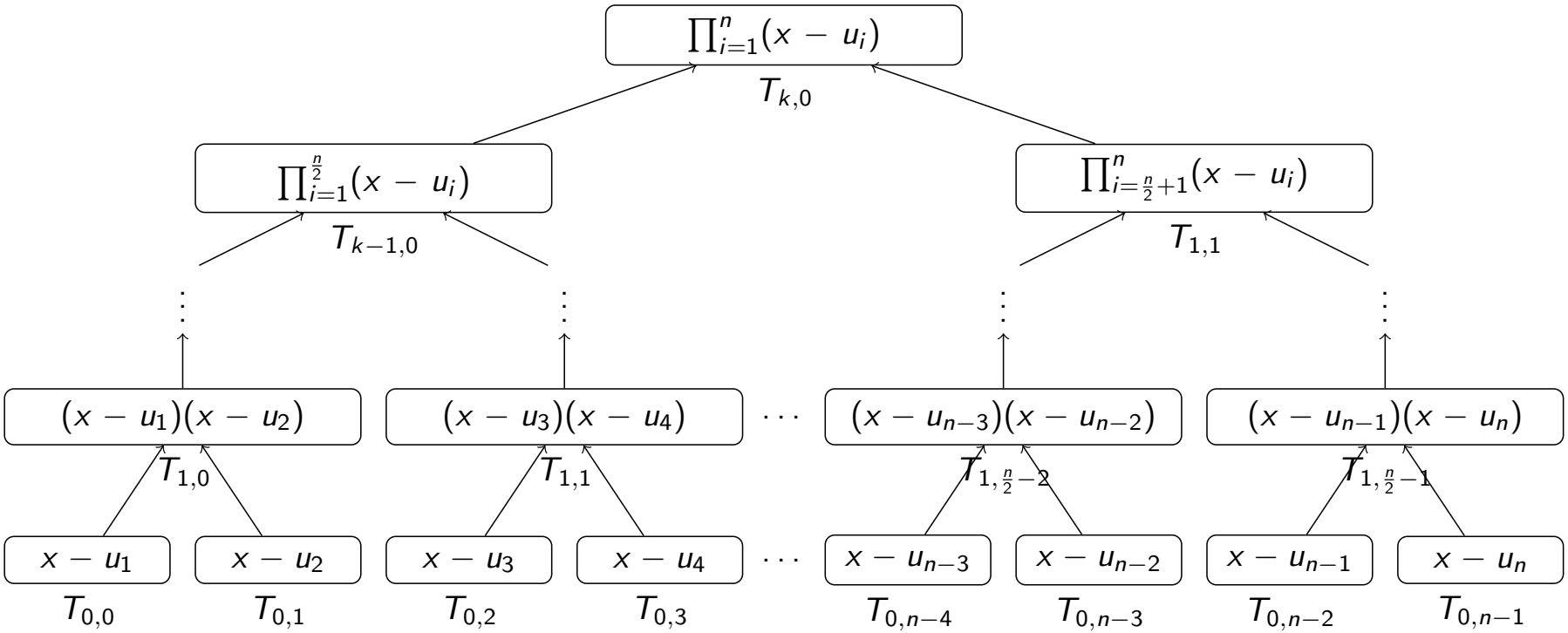
$$\prod_{i=1}^{3} p_i > (p-1)^2 \min(1 + \deg(f), 1 + \deg(g)).$$

- Use the FFT to multiply $f \times g \mod p_i$ for $i = 1, 2, 3$.
- Recover the integer coefficients in $f \times g$ before reduction mod p by Chinese remaindering.

Fast multiplication does $M(n) = 27 n \log_2 n + O(n)$ field operations in $\mathbb{Z}_p[x]$ [5].

Product Tree

Kaltofen and Yagati's method utilized Borodin and Munro's product tree [1] for polynomial multipoint evaluation algorithm with distinct evaluation points $u_1, u_2, \ldots, u_n \in F$ where $n = 2^k$ for some $k \in \mathbb{N}$.



Newton Inversion

Let $f, g \in \mathbb{Z}_p[x]$ such that $\deg(f) = m < 2n$ and $\deg(g) = n$ to compute r, q satisfying $f = g \cdot q + r$ where $\deg(r) < \deg(g)$ or $r = 0$

Newton inversion finds the inverse of the divisor polynomial to an order $m - n + 1$ approximation, i.e. $g^{-1} \mod x^{m-n+1}$.

Theorem (Theorem 9.2 [2])

Assume $h = \sum_{i=0}^{n} h_i x^i \in \mathbb{Z}_p[x]$ *and* $h_0 \neq 0$. *Let*

$$\begin{cases} y_0 = h_0^{-1} \\ y_i = 2y_{i-1} - h \cdot y_{i-1}^2 \mod x^{2^i} \quad \forall i \geq 1 \end{cases}$$

Then, for all $i \geq 0$,

$$h \cdot y_i \mod x^{2^i} = 1.$$

Newton inversion does at most $3M(n) + O(n)$ field operations in $\mathbb{Z}_p[x]$ when $m = 2n - 1$.

Newton Inversion with the Middle Product

In 2004, Hanrot, Quercia, and Zimmerman introduced an alternative formula for Newton inversion [3].

$$y_k = 2y_{k-1} - h \cdot y_{k-1}^2 = y_{k-1} + y_{k-1} \cdot (1 - h \cdot y_{k-1} \mod x^{2^k}) \mod x^{2^k}$$

Assuming $n = 2^k$, $h \cdot y_{k-1} \mod x^{\frac{n}{2}} = 1$. Then

$$h \cdot y_{k-1} \bmod x^n = 1 + \underbrace{m_0 x^{\frac{n}{2}} + \cdots + m_{\frac{n}{2}-1} x^{n-1}}_{= mp \cdot x^{\frac{n}{2}}} + \underbrace{a_0 x^n + \cdots + a_{\frac{n}{2}-2} x^{\frac{3n}{2}-2}}_{= a \cdot x^n} \bmod x^n$$

$$= 1 + mp \cdot x^{\frac{n}{2}}$$

The polynomial mp is called the middle product. Thus,

$$y_k = y_{k-1} + (y_{k-1} \cdot (-mp)) \cdot x^{\frac{n}{2}} \mod x^n.$$

Using the FFT of size n, Newton inversion costs at most $2M(n) + O(n)$.

Computing the FFT on y_{k-1} once reduces the cost to $\frac{5}{3}M(n) + O(n)$.

Fast Division

Definition

Let $f \in \mathbb{Z}_p[x]$ be a polynomial such that $f = f_0 + f_1 x + \cdots + f_{d-1} x^{d-1} + f_d x^d$ with $f_d \neq 0$. The reciprocal polynomial of f is

$$\widehat{f}(x) = x^d f\left(\frac{1}{x}\right) = f_0 x^n + f_1 x^{d-1} + \cdots + f_{d-1} x + f_d.$$

Theorem

Let $\mathbb{Z}_p$ be a field. Suppose $f, g \in \mathbb{Z}_p[x]$ are polynomials such that $\deg(f) = m$ and $\deg(g) = n$, where $m \geq n$. Let $r, q \in \mathbb{Z}_p[x]$ satisfy $f = g \cdot q + r$ where $r = 0$ or $\deg(r) < \deg(g)$. Then,

$$\widehat{q} = \widehat{f} \cdot (\widehat{g})^{-1} \mod x^{m-n+1}$$

Fast Division Algorithm

Input: Polynomials $f, g \in F[x]$ where $g \neq 0$.
Output: The remainder and quotient $r, q \in F[x]$ satisfying $f = g \cdot q + r$ where $r = 0$ or $\deg(r) < \deg(g)$

1: $m \leftarrow \deg(f)$
2: $n \leftarrow \deg(g)$
3: **if** $m < n$ **then return** $f, 0$ **end if**
4: $s \leftarrow m - n + 1$ // $\deg(q) = m - n$
5: $c \leftarrow \text{NIwithMP}(\widehat{g} \mod x^s, s)$ // $c = (\widehat{g})^{-1} \mod x^s$ ..... $I(n) \leq \frac{5}{3}M(n) + O(n)$
6: $e \leftarrow (\widehat{f} \mod x^s) \cdot c$ using Fast multiplication ..... $M(n)$
7: $\widehat{q} \leftarrow e \mod x^s$ // $\widehat{q} = \sum_{i=0}^{m-n} q_i^* x^i$
8: $q \leftarrow \sum_{i=0}^{m-n} q_{m-n-i}^* x^i$
9: $M \leftarrow g \cdot q$ using Fast multiplication ..... $M(n)$
10: $r \leftarrow f - M$
11: **return** r, q

This algorithm does at most $\frac{11}{3}M(n) + O(n)$ arithmetic operations in F when $\deg(f) = 2n - 1$.

Building the Product Tree

Let $f \in F[x]$ such that $\deg(f) = n - 1 < n$. Assume we have n distinct evaluation points $u_0, u_1, \ldots, u_{n-1}$.

Borodin and Munro presented the product tree

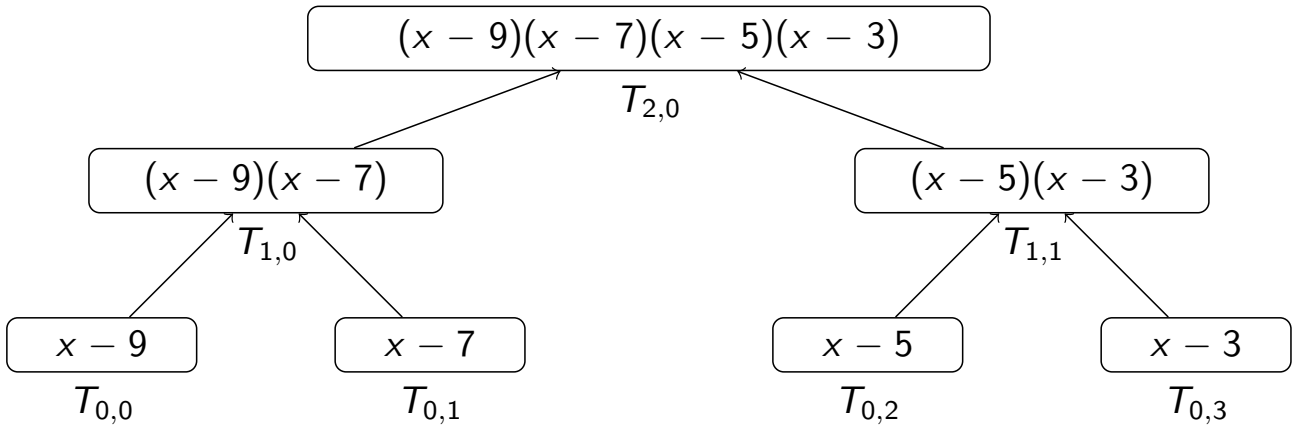


Figure: The product tree with evaluation points 9,7,5, and 3 in $\mathbb{Z}_{97}[x]$

Building the product tree (BuPT) does $M(n)\log_2 n + O(n \log n)$ field operations in $\mathbb{Z}_p$.

Optimizing the FFT, the cost reduces to $\frac{1}{2}M(n)\log_2 n + O(n \log n)$ field operations in $\mathbb{Z}_p$.

Dividing Down the Product Tree

Let $f = 1 + 2x + 3x^2 + 4x^3 \in \mathbf{Z}_{97}[x]$ to be evaluated.

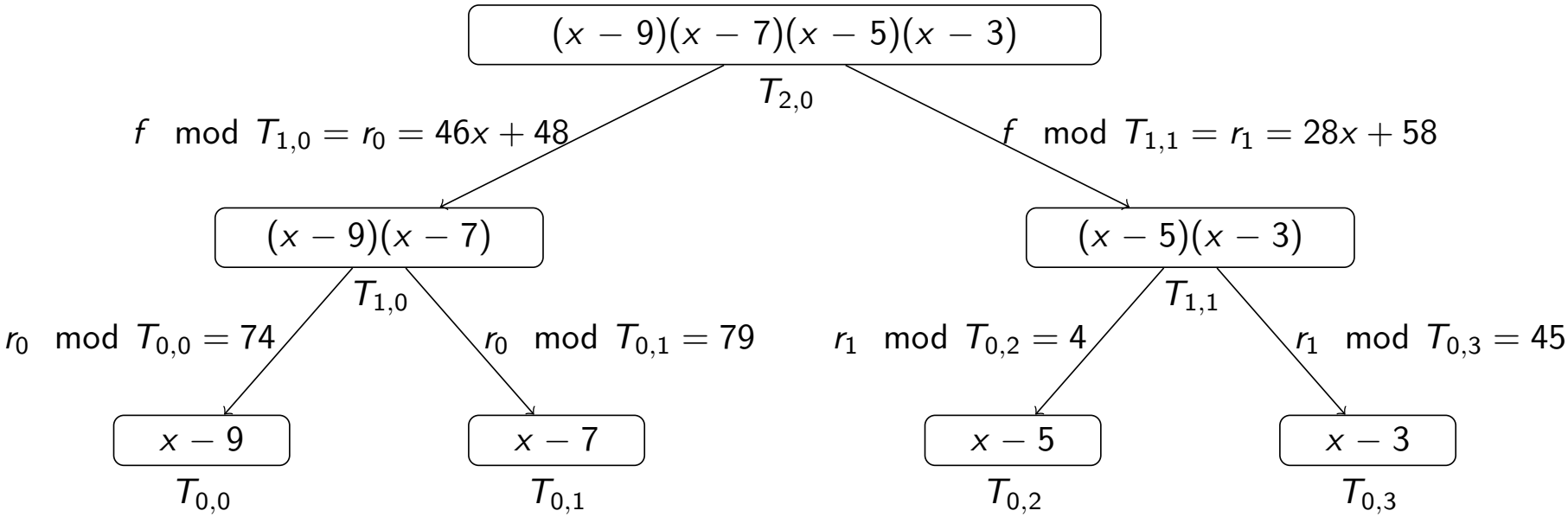


Figure: Dividing down the product tree to evaluate $f = 1 + 2x + 3x^2 + 4x^3 \in \mathbf{Z}_{97}[x]$

$f(9) = 74$, $f(7) = 79$, $f(5) = 4$, and $f(3) = 45$.

Dividing down the product tree (DDPT) does $\frac{11}{3} M(n) \log_2 n + O(n \log n)$ field operations in $\mathbb{Z}_p$.

Fast Multipoint Evaluation Algorithm

Input: $n = 2^k$ for some $k \in \mathbb{N}$, $f \in F[x]$ of degree less than n, and $u_0, u_1, \ldots, u_{n-1} \in F$.
Output: $f(u_0), f(u_1), \ldots, f(u_{n-1}) \in F$

1: $T \leftarrow \mathrm{BUPT}(n, u_0, u_1, \ldots, u_{n-1})$ ..... $B(n) \leq \frac{1}{2} M(n) \log_2 n + O(n \log n)$
2: $f(u_0), f(u_1), \ldots, f(u_{n-1}) \leftarrow \mathrm{DDPT}(n, f, T)$ ..... $C(n) \leq \frac{11}{3} M(n) \log_2 n + O(n \log n)$
3: **return** $f(u_0), f(u_1), \ldots, f(u_{n-1})$

This algorithm does at most $\frac{25}{6} M(n) \log_2 n + O(n \log n)$ field operations in $\mathbb{Z}_p$.

Remark: We can observe that DDPT is $\frac{22}{3}$ times as expensive as BUPT.

Zippel's Transposed Vandermonde Solver

Recall the following $n \times n$ transposed Vandermonde system of equations

$$\underbrace{\begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_1 & u_2 & \cdots & u_n \\ \vdots & \vdots & & \vdots \\ u_1^{n-1} & u_2^{n-1} & \cdots & u_n^{n-1} \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{\mathbf{b}}$$

In Zippel's method, define U^{-1}

$$U^{-1} = \begin{bmatrix} s_{1,1} & s_{1,2} & \cdots & s_{1,n} \\ s_{2,1} & s_{2,2} & \cdots & s_{2,n} \\ \vdots & \vdots & & \vdots \\ s_{n,1} & s_{n,2} & \cdots & s_{n,n}. \end{bmatrix}$$

Let $p_i(x) = s_{i,1} + s_{i,2}x + \cdots + s_{i,n}x^{n-1}$.

Zippel's Transposed Vandermonde Solver (continued)

Since $U^{-1}U = I$ and I is the $n \times n$ identity matrix,

$$\begin{bmatrix} s_{i,1} & s_{i,2} & \cdots & s_{i,n} \end{bmatrix} \begin{bmatrix} 1 \\ u_j \\ \vdots \\ u_j^{n-1} \end{bmatrix} = p_i(u_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Define $M = \prod_{i=1}^{n}(x - u_i)$ and $q_i(x) = M/(x - u_i)$.

Set $p_i(x) = q_i(u_i)^{-1} \cdot q_i(x)$ to recover i-th row entries of U^{-1}.

Compute the dot product of i-th row of U^{-1} and $\mathbf{b}$ to solve the transposed Vandermonde system.

This method does $O(n^2)$ field operations in $\mathbb{Z}_p$ and uses space for $O(n)$ elements of $\mathbb{Z}_p$.

Kaltofen & Yagati's Fast Transposed Vandermonde Solver

Since $p_i(x) = q_i(u_i)^{-1} \cdot q_i(x)$, $M = q_i(u_i) \cdot (x - u_i) \cdot p_i(x)$.

Define $D = b_n x + b_{n-1} x^2 + \cdots + b_1 x^n$. Let $H = M \cdot D = \sum_{i=0}^{2n-1} h_i x^{i+1}$.

$$H/(x - u_i) = q_i(u_i) \cdot p_i(x) \cdot D.$$

The coefficient of x^n in $H/(x - u_i)$ is

$$q_i(u_i) \cdot (s_{i,1} \cdot b_1 + s_{i,2} \cdot b_2 + \cdots + s_{i,n} \cdot b_n) = q_i(u_i) \cdot a_{i-1}.$$

In $H/(x - z)$, the coefficient of x^n is

$$v(z) = h_n + h_{n+1} z + \cdots + h_{2n-1} z^{n-1}.$$

Then $v(u_i) = q_i(u_i) \cdot a_{i-1}$. Also,

$$M'(x) = (x - u_i) \cdot q_i'(x) + q_i(x)$$

so $M'(u_i) = q_i(u_i)$. Thus, $a_{i-1} = v(u_i)/M'(u_i)$.

Fast Transposed Vandermonde Solver Algorithm

Input: $n = 2^k$ for some $k \in \mathbb{N}$, $u_1, u_2, \ldots, u_n \in \mathbb{Z}_p$, which compose the transposed Vandermonde matrix U and $\mathbf{b} = [b_1, b_2, \ldots, b_n] \in \mathbb{Z}_p^n$

Output: $\mathbf{a} = [a_0, a_1, \ldots, a_{n-1}] \in \mathbb{Z}_p^n$ satisfying $U\mathbf{a} = \mathbf{b}$

1: $T \leftarrow \text{BUPT}(n,\ u_1, u_2, \ldots, u_n)$ ..... $B(n) \leq \frac{1}{2}M(n)\log_2 n + O(n \log n)$
2: $M \leftarrow T_{k,0}$ from T
3: $D \leftarrow b_n x + b_{n-1}x^2 + \cdots + b_1 x^n$
4: $H \leftarrow M \cdot D$ using fast multiplication // $H = \sum_{i=0}^{2n-1} h_i x^{i+1}$ ..... $M(n)$
5: $Q \leftarrow \sum_{i=0}^{n-1} h_{n+i} z^i$ // Q = the coefficient of x^n in $H/(x - z)$
6: $q_1, q_2, \ldots, q_n \leftarrow \text{DDPT}(n, Q, T)$ // $q_i = Q(u_i)$ .. $C(n) \leq \frac{11}{3}M(n)\log_2 n + O(n \log n)$
7: Differentiate M
8: $r_1, r_2, \ldots, r_n \leftarrow \text{DDPT}(n, M', T)$ // $r_i = M'(u_i)$. $C(n) \leq \frac{11}{3}M(n)\log_2 n + O(n \log n)$
9: **for** i from 1 to n **do**
10: $\quad t \leftarrow r_i^{-1}$
11: $\quad a_{i-1} \leftarrow t \cdot q_i$
12: **end for**
13: **return** $[a_0, a_1, \ldots, a_{n-1}]$

This algorithm does $\frac{53}{6}M(n)\log_2 n + O(n \log n) \in O(M(n) \log n)$ field operations in $\mathbb{Z}_p$.

Case 1: $p = 116 \cdot 2^{55} + 1$

n	FastTVS					ZippelTVS	speedup	Maple	speedup
	BuPT	InvTree	DDPT1	DDPT2	Total				
2^6	0.046	-	0.046	0.039	0.195	0.1389	0.71	3.4	17.4
2^7	0.086	-	0.107	0.098	0.380	0.4879	1.28	8.6	22.6
2^8	0.150	-	0.254	0.238	0.808	1.9039	2.35	20.8	25.7
2^9	0.363	-	0.693	0.674	2.065	7.4640	3.61	63.0	30.5
2^{10}	0.875	0.600	1.890	1.877	5.811	30.826	5.30	113.2	19.5
2^{11}	2.020	2.417	5.070	5.008	15.775	116.84	7.40	270.0	17.1
2^{12}	4.755	7.529	12.307	12.268	39.444	469.64	11.90	608.0	15.4
2^{13}	11.146	20.556	29.566	29.270	95.765	1,868	19.50	1,321	13.8
2^{14}	25.901	53.099	71.091	70.580	231.55	7,456	32.19	3,025	13.1
2^{15}	60.151	131.30	166.15	166.46	546.52	29,986	54.86	7,190	13.2
2^{16}	131.23	314.56	380.02	376.77	1,249.3	120,292	96.28	16,455	13.2
2^{17}	339.89	746.70	867.30	863.48	2,914.9	478,912	164.3	69,705	23.9
2^{18}	663.01	1,747.1	1,961.8	1,955.2	6,529.8	1,929,776	295.5	97,667	15.0

Table: CPU timings in *ms* for solving $n \times n$ transposed Vandermonde systems over the prime field $\mathbb{Z}_p$ with $p = 116 \cdot 2^{55} + 1$

Case 2: $p = 14411518807585859$

n	FastTVS					ZippelTVS	speed up
	BuPT	InvTree	DDPT1	DDPT2	total		
2^6	0.043	-	0.042	0.040	0.247	0.1509	0.61
2^7	0.059	-	0.145	0.144	0.545	0.4869	0.89
2^8	0.276	-	0.343	0.340	1.355	1.9060	1.40
2^9	0.899	-	0.857	0.843	3.463	7.4809	2.16
2^{10}	2.615	-	2.388	2.398	9.195	29.702	3.23
2^{11}	6.685	-	7.510	7.374	25.353	116.44	4.59
2^{12}	16.044	-	25.265	25.139	73.946	470.04	6.35
2^{13}	38.754	80.014	76.846	76.688	288.00	1865.1	6.47
2^{14}	93.628	212.80	213.94	214.83	768.44	7478.1	9.73
2^{15}	214.23	510.61	541.49	540.96	1,875.7	29,763	15.86
2^{16}	497.72	1,237.4	1,343.2	1,354.4	4,576.6	119,478	26.10
2^{17}	1,111.9	2,890.1	3,199.1	3,210.3	10,716	488,369	45.57
2^{18}	2,494.2	6,632.9	7,480.6	7,470.6	24,725	1,953,252	78.99

Table: CPU timings in *ms* for solving $n \times n$ transposed Vandermonde system over $\mathbb{Z}_p$ with $p = 14411518807585859 < 2^{57}$

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