A Modular Algorithm to Compute the Resultant of Multivariate Polynomials over Algebraic Number Fields Presented with Multiple Extensions

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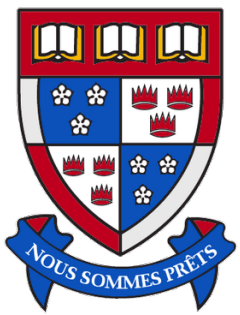

Background

Algebraic Number Field

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. The **algebraic number field** $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ is the smallest field containing $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_n$.

Theorem (1)

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. There exists $\gamma \in \mathbb{C}$ s.t

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \cong \mathbb{Q}(\gamma).$$

*We call γ a **primitive element**.*

Example

$\mathbb{Q}(\sqrt{2}, \sqrt{3}) \cong \mathbb{Q}(\sqrt{2} + \sqrt{3})$.

Resultant

Let $f_1, f_2 \in R[x_1, \ldots, x_k][y]$ and $f_1 = \sum_{j=0}^{m} a_j y^j$, $f_2 = \sum_{j=0}^{n} b_j y^j$.

$$\text{sylv}(f_1, f_2, y) = \begin{bmatrix} a_m & \cdots & a_0 & & \\ & a_m & \cdots & a_0 & \\ & & \ddots & & \\ & & a_m & \cdots & a_0 \\ b_n & \cdots & b_0 & & \\ & & \ddots & & \\ & & b_n & \cdots & b_0 \end{bmatrix}$$

Resultant

The resultant of f_1 and f_2 w.r.t. the variable y is defined as

$$res(f_1, f_2, y) = \det(\text{sylv}(f_1, f_2, y)) \in R[x_1, \ldots, x_k].$$

Why computing resultants is important?

Resultants appear as a subproblem in

- Factorization of polynomials over algebraic fields using Trager's algorithm.
- Solving systems of multivariate polynomials.
- Elimination theory.

History

1971, Collins

A modular algorithm to compute the resultant of multivariate polynomials over $\mathbb{Z}$.

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2002-Monagan and Van Hoeij

A modular GCD algorithm for polynomials in $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)[x]$.

History

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2023-Ansari and Monagan

A modular GCD algorithm that reduces the gcd problem over $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ to gcd calculation over $\mathbb{Q}(\gamma)$ where γ is a primitive element of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.

Our Contributions

Let $f_1, f_2 \in \mathbb{Q}(\alpha_1, \ldots, \alpha_n)[x_1, \ldots, x_k, y]$.

1. $MRES(f_1, f_2, y)$ computes $r = \text{res}(f_1, f_2, y)$.
2. Employing the monic Euclidean algorithm for $f_1, f_2 \in \mathbb{Z}_p[\alpha_1, \ldots, \alpha_n][y]$, we present a new formula for computing $\text{res}(f_1, f_2, y)$.
3. A Maple implementation by using RECDEN, a Maple library for recursive dense representation for polynomials.
4. The expected time complexity analysis.
5. A partial failure probability analysis.

Computation over $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$

Let

$$\begin{aligned} L_0 &= \mathbb{Q} \\ L_i &= L_{i-1}/\langle M_i(z_i)\rangle \quad for\ 1 \leq i \leq n \\ L_n &= L \end{aligned}$$

where $M_i(z_i)$ is the minimal polynomial of α_i over L_{i-1}.

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \cong L$$

Remark

MRES assumes that we are given the minimal polynomials $M_i(z_i)$.

In order to improve computational efficiency, in a preprocessing step in MRES, we eliminate fractions from the input polynomials f_1 and f_2 and the minimal polynomials $M_1, \ldots, M_n$.

Example

Let $f = \frac{3}{2}\alpha_1 x + \alpha_2 \in \mathbb{Q}(\alpha_1, \alpha_2)[x]$ where $\alpha_1 = \sqrt{2}$ and $\alpha_2 = \sqrt{3}$. Eliminate fractions from f, we have

$$\check{f} = 3\alpha_1 x + 2\alpha_2.$$

Monic Euclidean Algorithm (MEA)

Algorithm 2: Monic Euclidean Algorithm

Input: $f_1, f_2 \in R[x]$ such that $0 \leq \deg(f_2) \leq \deg(f_1)$ and R is a commutative ring with identity $1 \neq 0$.

Output: Either the monic $\gcd(f_1, f_2)$ or FAIL.

1 $r_1, r_2 = f_1, f_2$
2 $M_1, i = r_1, 2$
3 **while** $r_i \neq 0$ **do**
4 $\quad M_i = monic(r_i)$
5 $\quad$ **if** $M_i = failed$ **then return** *(FAIL)* `// The algorithm encountered a zero-divisor.`
6 $\quad$ Set r_{i+1} to be the remainder of M_{i-1} divided by M_i
7 $\quad$ Set $i = i + 1$
8 $l = i - 1$
9 **return**(M_l)

Note:

MEA fails if a remainder r_i exists, such that $\mathrm{lc}(r_i)$ is not invertible over R. In other words, if $M_i = (r_i)$ fails at Step 4, then MEA returns FAIL.

m.p.r.s.

Given $f_1, f_2 \in R[x]$ with $\deg(f_2) \leq \deg(f_1)$, assume that the Monic Euclidean Algorithm (MEA) does not fail for f_1 and f_2 and terminates after l iterations. We define the Monic Polynomial Remainder Sequence, m.p.r.s., generated by polynomials f_1 and f_2 as the sequence $r_1, r_2, \ldots, r_l, r_{l+1}$ obtained from the execution of the Monic Euclidean Algorithm such that $r_1 = f_1$, $r_2 = f_2$, $r_3 = r_1 - M_2 q_3$, and $r_{i+1} = M_{i-1} - M_i q_{i+1}$ with $M_i = monic(r_i)$ and $\deg(r_{i+1}) < \deg(r_i)$ for $2 \leq i \leq l-1$ and $r_{l+1} = 0$.

Using MEA to Compute Resultant

Theorem

Suppose that $f_1, f_2 \in R[x]$ *and the Monic Euclidean Algorithm does not fail for* f_1 *and* f_2. *Let* $r_1, r_2, \ldots, r_l, r_{l+1}$ *be the m.p.r.s. generated by* f_1 *and* f_2 *where* $r_{l+1} = 0$. *Let* $n_i = \deg(r_i)$ *for* $1 \leq i \leq l$. *If* $\deg(r_l) > 0$, *then* $\operatorname{res}(f_1, f_2) = 0$. *Otherwise, we have*

$$\operatorname{res}(f_1, f_2) = (-1)^v \Big(\prod_{i=2}^{l-1} \operatorname{lc}(r_i)^{n_{i-1}}\Big) \operatorname{lc}(r_l)^{n_{l-1}}$$

where $v = \sum_{i=1}^{l-2} n_i n_{i+1}$.

We can modify the MEA to compute the resultant of two univariate polynomials $f_1, f_2 \in R[x]$ in algorithm URES.

Algorithm 3: URES

Require: $f_1, f_2 \in R[x]$ such that $0 \leq \deg(f_2) \leq \deg(f_1)$ where R is a commutative ring with identity $1 \neq 0$.

Ensure: Either $\text{res}(f_1, f_2)$ or FAIL.

1: $r_1 = f_1$, $r_2 = f_2$, $i = 2$
2: $M_1 = r_1$, $R = 1$, $v = 0$
3: $n_1 = \deg(f_1)$,$n_2 = \deg(f_2)$
4: **while** $r_i \neq 0$ **do**
5: $M_i = monic(r_i)$
6: **if** $M_i = failed$ **return** *(FAIL)*// The algorithm encounters a zero-divisor.
7: Set r_{i+1} to be the remainder of M_{i-1} divided by M_i
8: Set $n_{i+1} = \deg(r_{i+1})$
9: **if** $n_{i+1} < 0$ and $n_i \neq 0$ **then return**(0) // If $\gcd(f_1, f_2)$ is not a constant, then $\text{res}(f_1, f_2) = 0$
10: Set $R = R \cdot \text{lc}(r_i)^{n_i - 1}$
11: Set $v = v + n_i n_{i-1}$
12: Set $i = i + 1$
13: **end while**
14: $R = (-1)^v R$
15: **return**(R)

MRES

Let $f_1, f_2 \in L[x_1, \ldots, x_k][y]$, MRES computes $\text{res}(f_1, f_2, y)$.

$$
\begin{array}{ccc}
\check{f}_1, \check{f}_2 \in L_{\mathbb{Z}}[x_1, \ldots, x_k][y] & \longrightarrow & \text{res}(f_1, f_2, y) \in L[x_1, \ldots, x_k] \\
\Big\downarrow {\scriptstyle \phi_p \text{ for } p \in \{p_1, p_2, \ldots\}} \; {\scriptstyle p \nmid \prod_{i=1}^{n} \text{lc}(\check{M}_i) \cdot \text{lc}(\check{f}_1)} & & \Big\uparrow {\scriptstyle CRT,\ RNR} \\
\check{f}_1, \check{f}_2 \in L_p[x_1, \ldots, x_k][y] & & \text{res}(\check{f}_1, \check{f}_2, y) \in L_p[x_1, \ldots, x_k] \\
\Big\downarrow {\scriptstyle \phi_\gamma} & & \Big\uparrow {\scriptstyle \phi_\gamma^{-1}} \\
\check{f}_1, \check{f}_2 \in \bar{L}_p[x_1, \ldots, x_k][y] & \xrightarrow{PRES} & \text{res}(\check{f}_1, \check{f}_2, y) \in \bar{L}_p[x_1, \ldots, x_k]
\end{array}
$$

$L = \mathbb{Q}[z_1, \ldots, z_n] / \langle M_1(z_1), \ldots, M_n(z_n) \rangle$
$L_{\mathbb{Z}} = \mathbb{Z}[\alpha_1, \ldots, \alpha_n]$
$L_p = \mathbb{Z}_p[z_1, \ldots, z_n] / \langle m_1(z_1), \ldots, m_n(z_n) \rangle$ s.t $m_i(z_i) = \check{M}_i(z_i) \mod p$
$\bar{L}_p = \mathbb{Z}_p[z] / \langle M(z) \rangle$

Reduction mod p

The modular homomorphism

$\phi_p : \mathbb{Z} \longrightarrow \mathbb{Z}_p$ maps integers into their remainder modulo p. We choose p to be a prime so $\mathbb{Z}_p$ is a finite field.

$$\begin{array}{ccc}
\check{f}_1, \check{f}_2 \in L_{\mathbb{Z}}[x_1, \ldots, x_k][y] & \longrightarrow & \mathrm{res}(f_1, f_2, y) \in L[x_1, \ldots, x_k] \\
\phi_p \text{ for } p \in \{p_1, p_2, \ldots\} \Big\downarrow p \nmid \prod_{i=1}^{n} \mathrm{lc}(\check{M}_i) \cdot \mathrm{lc}(\check{f}_1) & & \Big\uparrow CRT, RNR \\
\check{f}_1, \check{f}_2 \in L_p[x_1, \ldots, x_k][y] & & \mathrm{res}(\check{f}_1, \check{f}_2, y) \in L_p[x_1, \ldots, x_k] \\
\Big\downarrow \phi_\gamma & & \Big\uparrow \phi_\gamma^{-1} \\
\check{f}_1, \check{f}_2 \in \bar{L}_p[x_1, \ldots, x_k][y] & \xrightarrow{PRES} & \mathrm{res}(\check{f}_1, \check{f}_2, y) \in \bar{L}_p[x_1, \ldots, x_k]
\end{array}$$

$$L_p = \mathbb{Z}_p[z_1, \ldots, z_n] / \langle m_1(z_1), \ldots, m_n(z_n) \rangle \text{ s.t } m_i(z_i) = \check{M}_i(z_i) \mod p$$

Lc-bad, Det-bad, and Zero-divisor

Question: Can we use any prime?

Answer: No!!!

Lc-bad, Det-bad, and Zero-divisor

Let $f_1, f_2 \in L[x_1, \ldots, x_k][y]$ and p be a prime.

- Lc-bad Prime: If p divides either $\mathrm{lc}(\check{f}_1)$, $\mathrm{lc}(\check{f}_2)$, or any $\mathrm{lc}(\check{M}_i(z_i))$ for $1 \leq i \leq n$, we call p an lc-bad prime.
- Det-bad Prime: Let B_{Lp} be a basis of L_p and γ be a primitive element and $A = \left[\left[\begin{matrix} \vdots \\ 1 \\ \vdots \end{matrix} \right]_{B_{Lp}} \quad \cdots \quad \left[\begin{matrix} \vdots \\ \gamma^{d-1} \\ \vdots \end{matrix} \right]_{B_{Lp}} \right]$.
  If $\det(A) \mod p = 0$, then p is called a det-bad prime.
- Zero-divisor Prime: If p is neither an lc-bad nor a det-bad prime and algorithm URES fails over $\bar{L}_p$, then p is called a zero-divisor prime.
- Good Prime: If p is neither lc-bad, det-bad nor a zero-divisor prime, we define it as a good prime.

Example

Let $L = \mathbb{Q}[z_1, z_2]/\langle z_1^2 - 2, z_2^2 - 3\rangle$ and

$$f_1 = 23z_2x + z_1y$$
$$f_2 = (z_2 + 5)x + z_1y$$

be two polynomials listed in the lexicographic order with $x > y$ over $L[x, y]$.

- $p_1 = 23 \Longrightarrow \phi_{23}(\mathrm{lc}(\check{f}_1)) = 0 \mod p_1$. Thus p_1 is an lc-bad prime.
- $p_2 = 11 \Longrightarrow \mathrm{lc}(f_2) = z_2 + 5$ and $z_2^2 - 3 \mod 11 = (z_2 + 5)(z_2 + 6)$ so $z_2 + 5$ is not invertible over $\mathbb{Z}_{11}[z_1, z_2]/\langle z_1^2 - 2, z_2^2 - 3\rangle$.

The isomorphism ϕ_γ

$\phi_\gamma : L_p[x_1, \ldots, x_k] \longrightarrow \bar{L}_p[x_1, \ldots, x_k]$ maps f over $\mathbb{Z}_p[z_1, \ldots, z_n]/\langle m_1(z_1), \ldots, m_n(z_n)\rangle$ to its corresponding polynomial over $\mathbb{Z}_p[z]/\langle M(z)\rangle$.

$$
\begin{array}{ccc}
\check{f}_1, \check{f}_2 \in L_{\mathbb{Z}}[x_1, \ldots, x_k][y] & \longrightarrow & \mathrm{res}(f_1, f_2, y) \in L[x_1, \ldots, x_k] \\
\Big\downarrow \phi_p \text{ for } p \in \{p_1, p_2, \ldots\} \; p \nmid \prod_{i=1}^{n} \mathrm{lc}(\check{M}_i) \cdot \mathrm{lc}(\check{f}_1) & & \Big\uparrow \mathit{CRT},\ \mathit{RNR} \\
\check{f}_1, \check{f}_2 \in L_p[x_1, \ldots, x_k][y] & & \mathrm{res}(\check{f}_1, \check{f}_2, y) \in L_p[x_1, \ldots, x_k] \\
\Big\downarrow \phi_\gamma & & \Big\uparrow \phi_\gamma^{-1} \\
\check{f}_1, \check{f}_2 \in \bar{L}_p[x_1, \ldots, x_k][y] & \xrightarrow{\mathit{PRES}} & \mathrm{res}(\check{f}_1, \check{f}_2, y) \in \bar{L}_p[x_1, \ldots, x_k]
\end{array}
$$

$L_p = \mathbb{Z}_p[z_1, \ldots, z_n]/\langle m_1(z_1), \ldots, m_n(z_n)\rangle$ s.t $m_i(z_i) = \check{M}_i(z_i) \mod p$
$\bar{L}_p = \mathbb{Z}_p[z]/\langle M(z)\rangle$

For more detailed information on how ϕ_γ works, please refer to our previous paper

[1] Ansari, Monagan
Computing GCDs of multivariate polynomi- als over algebraic number fields presented with multiple extensions.
Computer Algebra in Scientific Computing, volume 14139 of LNCS, page 1–20. Springer, 2023. .

PRES

Algorithm PRES is a recursive algorithm to compute the $\text{res}(f_1, f_2, y)$ where f_1, f_2 belong to $\bar{L}_p[x_1, \ldots, x_k][y]$ and p is a prime.

$$
\begin{array}{ccc}
\check{f}_1, \check{f}_2 \in L_{\mathbb{Z}}[x_1, \ldots, x_k][y] & \longrightarrow & \text{res}(f_1, f_2, y) \in L[x_1, \ldots, x_k] \\
\Big\downarrow \phi_p \text{ for } p \in \{p_1, p_2, \ldots\},\ p \nmid \prod_{i=1}^{n} \text{lc}(\check{M}_i) \cdot \text{lc}(\check{f}_1) & & \Big\uparrow \textit{CRT, RNR} \\
\check{f}_1, \check{f}_2 \in L_p[x_1, \ldots, x_k][y] & & \text{res}(\check{f}_1, \check{f}_2, y) \in L_p[x_1, \ldots, x_k] \\
\Big\downarrow \phi_\gamma & & \Big\uparrow \phi_\gamma^{-1} \\
\check{f}_1, \check{f}_2 \in \bar{L}_p[x_1, \ldots, x_k][y] & \xrightarrow{\textit{PRES}} & \text{res}(\check{f}_1, \check{f}_2, y) \in \bar{L}_p[x_1, \ldots, x_k]
\end{array}
$$

1. If $k = 0$, then PRES calls the algorithm URES to compute $\text{res}(f_1, f_2, y) \in \bar{L}_p$. PRES might fail.
2. PRES uses dense evaluation and interpolation to recover $x_1, \ldots, x_k$.

Question: Can we use any evaluation point?

Answer: No!!!

Benchmark 1

Let $L = \mathbb{Q}(\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7})$ have degree 16. the input polynomials f_1 and f_2 have degree m in x and y and $\text{res}(f_1, f_2, x)$ has degree r_y in y. The coefficients of the input polynomials, f_1 and f_2, are polynomials in z_1, z_2, z_3, and z_4 with coefficients chosen randomly from $[1, 9)$.

m	r_y	N	MRES 1			MRES 2	
			time	ϕ_γ	PRES	time	PRES
2	4	4	0.313	0.126	0.140	0.828	0.828
4	16	4	0.828	0.187	0.501	8.609	8.563
6	36	7	3.938	0.189	3.218	59.938	59.610
8	64	11	14.171	0.218	11.891	291.281	289.875
10	100	16	47.500	0.468	48.842	967.437	962.609
12	144	22	119.766	0.596	103.016	> 1000	> 1000
14	196	29	282.844	0.798	244.189	> 1000	> 1000

Benchmark 2

Let f_1 and f_2 be two polynomials in $L[x, y]$, where $L = \mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$. Let $M_1 = z_1^2 - 2$, $M_2 = z_2^2 - 3$, and $M_3 = \sum_{j=0}^{d_3} z_3^j + z_1 z_2$ be the minimal polynomials of α_1, α_2, and α_3, respectively. Thus, L is an algebraic number field of degree $d = 2 \times 2 \times d_3$. To consider various degrees for L, we change d_3.

d	N	MRES 1			MRES 2	
		time	ϕ_γ	PRES	time	PRES
16	5	43.688	0.095	42.562	288.265	287.811
24	5	55.203	0.156	53.688	379.735	379.077
32	5	57.234	0.249	55.517	513.797	513.078
40	5	67.719	0.375	65.641	628.547	627.361
48	5	80.687	0.624	78.094	745.578	744.703
56	5	100.953	0.922	97.031	894.734	893.921
64	5	114.062	1.375	110.171	> 1000	> 1000

Complexity

Let $f_1, f_2 \in L[x_1, \cdots, x_k, y]$.

1. Let d be the degree of the number field L.
2. $T_f = \#f_1 + \#f_2$
3. $m = \deg(f_1, y)$, $n = \deg(f_2, y)$.
4. $D = (m+n)dx$ where $dx = \max_{i,j} \deg(f_i, x_j)$

We assume that multiplication and inverses in $\bar{L}_p$ cost $O(d^2)$.

Theorem

Algorithm PRES does

$$O(dD^k(T_f + mnd + kD)$$

arithmetic operations in $\mathbb{Z}_p$.

Let $r = \text{res}(f_1, f_2, y)$ and

- $T_r = \#r$
- N is the number of good primes needed to reconstruct the resultant r.
- $M = \log \max_{i=1}^{n} H(\check{m}_i)$.
- $C = \log \max(H(\check{f}_1), H(\check{f}_2))$.

Theorem

Algorithm MRES does

$$O(Nd(M + CT_M + d^2 + d(T_f + T_r) + D^k(T_f + mnd + kD) + NT_r))$$

arithmetic operations.

Failure probability of lc-bad primes

Let $\mathbb{P}_{31} = \{\text{all 31 bit primes}\}$.
We have $|\ \mathbb{P}_{31}\ | = 50,697,537$.

Theorem

Let $f_1, f_2 \in L[x_1, \ldots, x_k, y]$ *and*

1. $H = \max(\|\mathrm{lc}(\check{f}_1)\|)_\infty, \|\mathrm{lc}(\check{f}_2)\|)_\infty) < 2^h$,
2. $\mathrm{lc}(\check{M}_i) < 2^m$ *for* $1 \le i \le n$.

If p *is chosen at random from* $\mathbb{P}_{31}$ *then*

$$\mathrm{Prob}[p \text{ is an lc-bad prime}] \le \frac{2\lfloor \frac{h}{30} \rfloor + n\lfloor \frac{m}{30} \rfloor}{|\ \mathbb{P}_{31}\ |}.$$

Failure probability of lc-bad evaluation point

Theorem

Let $\beta \in \mathbb{Z}_p^k$ be chosen at random, then

$$\text{Prob}[\beta \textit{ is an lc-bad evaluation point}] \leq \frac{\deg(\check{f}_1)}{p}.$$

Failure probability of Det-bad Primes

In our paper we considered the case where $\check{m}_i \in \mathbb{Z}[z_1, \ldots, z_i]$ are monic for $1 \leq i \leq n$ so $A \in \mathbb{Z}^{d \times d}$.

Theorem

Let $\gamma = z_1 + C_1 z_2 + \cdots + C_{n-1} z_n$ *where* $0 \neq C_i \in \mathbb{Z}$ *for* $1 \leq i \leq n-1$ *and* $\|\gamma^d\|_\infty \leq 2^C$. *Suppose* $\det(A) \neq 0$. *If* p *is chosen at random from* $\mathbb{P}_{31}$ *then*

$$\text{Prob}[p | \det(A)] \leq \frac{\lfloor (d/2 \log_2 d + d(C + \sum_{i=1}^{n} \delta_i \log_2(1 + D_i \|\check{m}_{n-i+1}\|_\infty))) \rfloor}{30 \mid \mathbb{P}_{31} \mid}.$$

where $D_i = \frac{d}{\prod_{j=1}^{i} d_{n-j+1}}$, $\delta_1 = d - d_n + 1$, *and*
$\delta_i = d - d_{n-i+1} + 1 + (d_{n-i+1} - 1) \sum_{j=1}^{i-1} \delta_j$ *for* $2 \leq i \leq n$.

Figura: Portrait of Dr. Gachet, Van Gogh, 1890, France

Future Work

Computing the failure probability of zero-divisor evaluation points and primes

Thank you!

Figura: Wanderer above sea fog, Caspar David Friedrich, 1818