Computing over Algebraic Number Fields Modulo Primes Using Linear Algebra

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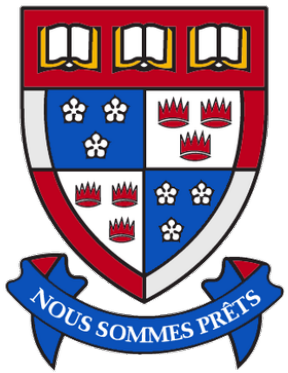

Outline

Algebraic Numbers

Algebraic Number

A complex number $\alpha \in \mathbb{C}$ is called an **Algebraic Number** if there exists a non-zero polynomial $M(z) \in \mathbb{Q}[z]$ s.t. $M(\alpha) = 0$.
For instance, $\alpha = \sqrt{2}$ is an algebraic number since it is a root of $M(z) = 2z^2 - 4$.

Minimal Polynomial

If α is an algebraic number, there is a unique non-zero polynomial $M(z) \in \mathbb{Q}[z]$ of least degree such that

1) $M(\alpha) = 0$
2) $M(z)$ is monic
3) $M(z)$ is irreducible.

We call $M(z)$ the **Minimal Polynomial** of α.
For example, $M(z) = z^2 - 2$ is the minimal polynomial of $\alpha = \sqrt{2}$.

Algebraic Number Fields

Algebraic Number Field

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. The **Algebraic Number Field** $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ is the smallest field containing $\mathbb{Q}$ and $\alpha_1, \ldots, \alpha_n$.

Question: How can we compute in $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$?

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Question: How can we compute in $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$?

Answer: Let $L_0 = \mathbb{Q}$

$$L_i = L_{i-1}[z_i]/\langle M_i(z_i)\rangle \quad for\ 1 \leq i \leq n$$
$$L_n = L$$

where $M_i(z_i)$ is the minimal polynomial of α_i over L_{i-1}. Then

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \cong L.$$

Map the elements of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$ to L and do the computation in L.

Notation:

We also write $L = \mathbb{Q}[z_1, \ldots, z_n]/\langle M_1, \ldots, M_n\rangle$.

- L can be specified as a $\mathbb{Q}-$vector space.
- Let $d_i = \deg(M_i(z_i), z_i)$ and $d = \prod_{i=1}^{n} d_i$. Then, $B_L = \{\prod_{i=1}^{n}(z_i)^{e_i} | \ 0 \leq e_i < d_i\}$ is a basis for L.
- The degree of L is $[L : \mathbb{Q}] =| \ B_L \ |= d$.

Example

Let $L = \mathbb{Q}[z_1, z_2]/\langle z_1^2 - 2, z_2^2 - 3\rangle$. Then $B_L = \{1, z_1, z_2, z_1 z_2\}$ is a basis for L and $[L : \mathbb{Q}] = 4$.

Primitive Element

Theorem

Let $\alpha_1, \ldots, \alpha_n$ be algebraic numbers. There exists an algebraic number γ s.t.

$$\mathbb{Q}(\alpha_1, \ldots, \alpha_n) \cong \mathbb{Q}(\gamma).$$

*We call γ a **Primitive Element**.*

Example

Let $\alpha_1 = \sqrt{2}$, $\alpha_2 = \sqrt{3}$. Then $\gamma = \sqrt{2} + \sqrt{3}$ is a primitive element of $\mathbb{Q}(\alpha_1, \alpha_2)$ and

$$\mathbb{Q}(\alpha_1, \alpha_2) \cong \mathbb{Q}(\gamma).$$

Question:

How can we speed up computation over $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$?

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Answer:

1. First, use a modular homomorphism. Let p be a prime and $L_p = \mathbb{Z}_p[z_1, \ldots, z_n]/\langle m_1, \ldots, m_n \rangle$.
2. Then, use the **primitive element isomorphism**

$$\phi_\gamma : \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \longrightarrow \mathbb{Q}(\gamma)$$

modulo p. Let $\overline{L}_p = \mathbb{Z}_p[z]/\langle M(z) \rangle$, where $M(z)$ is the minimal polynomial of γ modulo p. Then

$$\phi_\gamma : L_p \longrightarrow \overline{L}_p.$$

Why?

Computation in $\overline{L}_p$ is more efficient than computation in L_p.

The Number of Multiplications (L_p)

Let p be a prime. Then

$$\begin{aligned} L_0 &= \mathbb{Z}_p \\ L_1 &= L_0[z_1]/\langle m_1(z_1)\rangle \quad \text{s.t.} \quad d_1 = \deg(m_1, z_1) = 2 \\ L_2 &= L_1[z_2]/\langle m_2(z_2)\rangle \quad \text{s.t.} \quad d_2 = \deg(m_2, z_2) = 4. \end{aligned}$$

Let $L_p = L_2$ and let $A, B \in L_p$ s.t.

$$A = \sum_{i=0}^{3} a_i(z_1) z_2^i, \quad \text{where} \quad a_i(z_1) = a_{i0} + a_{i1} z_1 \in L_1, \quad \text{and}$$

$$B = \sum_{i=0}^{3} b_i(z_1) z_2^i, \quad \text{where} \quad b_i(z_1) = b_{i0} + b_{i1} z_1 \in L_1.$$

In the recursive dense representation, using the `recden` library in Maple, $A, B \in L_p$ are stored as nested lists:

$$A = [\underbrace{[a_{00}, a_{01}]}_{z_2^0}, \underbrace{[a_{10}, a_{11}]}_{z_2^1}, \underbrace{[a_{20}, a_{21}]}_{z_2^2}, \underbrace{[a_{30}, a_{31}]}_{z_2^3}]$$

$$B = [\underbrace{[b_{00}, b_{01}]}_{z_2^0}, \underbrace{[b_{10}, b_{11}]}_{z_2^1}, \underbrace{[b_{20}, b_{21}]}_{z_2^2}, \underbrace{[b_{30}, b_{31}]}_{z_2^3}].$$

$A \times B$ requires:

- $d_1^2 = 4$ $\times$ in $\mathbb{Z}_p$ before reduction modulo M_1. Reduction modulo M_1 costs $d_1(d_1 - 1) = 2$ additional $\times$ in $\mathbb{Z}_p$. Therefore, one $\times$ in L_1 requires at most $d_1^2 + d_1(d_1 - 1) = 6$ $\times$ in $\mathbb{Z}_p$.
- At the next level, $A \times B$ as polynomials in z_2 requires $d_2^2 = 16$ coefficient $\times$. Similarly, $\times$ and reduction at the z_2-level cost $d_2^2 + d_2(d_2 - 1) = 28$ $\times$ in L_1.
- Therefore, $A \times B$ requires $6 \cdot 28 = 168$ $\times$ in $\mathbb{Z}_p$.

The Number of Multiplications ($\overline{L}_p$)

Let $\overline{L}_p = \mathbb{Z}_p[z]/\langle M(z)\rangle \cong L_p$, where $\deg(M, z) = 8$. Let $A_\gamma, B_\gamma \in \overline{L}_p$:

$$A_\gamma = \sum_{i=0}^{7} a_i z^i, \qquad B_\gamma = \sum_{i=0}^{7} b_i z^i, \text{ with } a_i, b_i \in \mathbb{Z}_p.$$

The data representation of A_γ and B_γ in recden is presented as

$$A_\gamma = [\underbrace{a_0}_{z^0}, \underbrace{a_1}_{z^1}, \underbrace{a_2}_{z^2}, \underbrace{a_3}_{z^3}, \underbrace{a_4}_{z^4}, \underbrace{a_5}_{z^5}, \underbrace{a_6}_{z^6}, \underbrace{a_7}_{z^7}]$$
$$B_\gamma = [\underbrace{b_0}_{z^0}, \underbrace{b_1}_{z^1}, \underbrace{b_2}_{z^2}, \underbrace{b_3}_{z^3}, \underbrace{b_4}_{z^4}, \underbrace{b_5}_{z^5}, \underbrace{b_6}_{z^6}, \underbrace{b_7}_{z^7}].$$

$A_\gamma \times B_\gamma$ costs $d^2 = 8^2 = 64$ $\times$ in $\mathbb{Z}_p$. The unreduced product has degree at most $2d - 2$. Reducing it modulo M requires at most $d(d-1) = 8 \cdot 7 = 56$ $\times$ in $\mathbb{Z}_p$. Therefore, $A_\gamma \times B_\gamma$ requires at most $d^2 + d(d-1) = 64 + 56 = 120$ $\times$ in $\mathbb{Z}_p$.

The Number of Storage Allocation

- **# Storage allocations (L_p):** A $\times$ in L_p does $d_2^2 + d_2(d_2 - 1) = 28$ $\times$ in L_1. Each needs at least 2 pieces of storage, one for a product and one for a remainder. Also, it requires two pieces of storage: one for the product in L_2 and one remainder in L_2. In total, it requires 58 pieces of storage.
- **# Storage allocations ($\overline{L}_p$):** In comparison, $\times$ in $\overline{L}_p$ needs only 2 pieces of storage, one for the product of size $2d$ and one for the remainder of size d.

In summary, we have

Ring	#mults in $\mathbb{Z}_p$	#storage allocations
L_p	168	58
$\overline{L}_p$	120	2

Table: # mults and #storage

Outline

Finding a primitive element γ and its minimal polynomial $M(z)$

Let $\gamma = z_1 + C_1 z_2 + \ldots + C_{n-1} z_n$ be a candidate primitive element, where $0 \neq C_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$.

Question: How can we verify whether γ is a primitive element of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$?

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Question: How can we verify whether γ is a primitive element of $\mathbb{Q}(\alpha_1, \ldots, \alpha_n)$?

Answer: There are several methods that we can use:

- **Using Gröbner Bases**: Requires $O(nd^3)$ arithmetic operations in $\mathbb{Q}$.
- **Using Resultants**: Does not directly provide the isomorphism $\phi_\gamma : \mathbb{Q}(\alpha_1, \ldots, \alpha_n) \longrightarrow \mathbb{Q}(\gamma)$.
- **Using Linear Algebra**: Requires $O(d^3)$ arithmetic operations in $\mathbb{Q}$.

Finding a primitive element γ and its minimal polynomial $M(z)$

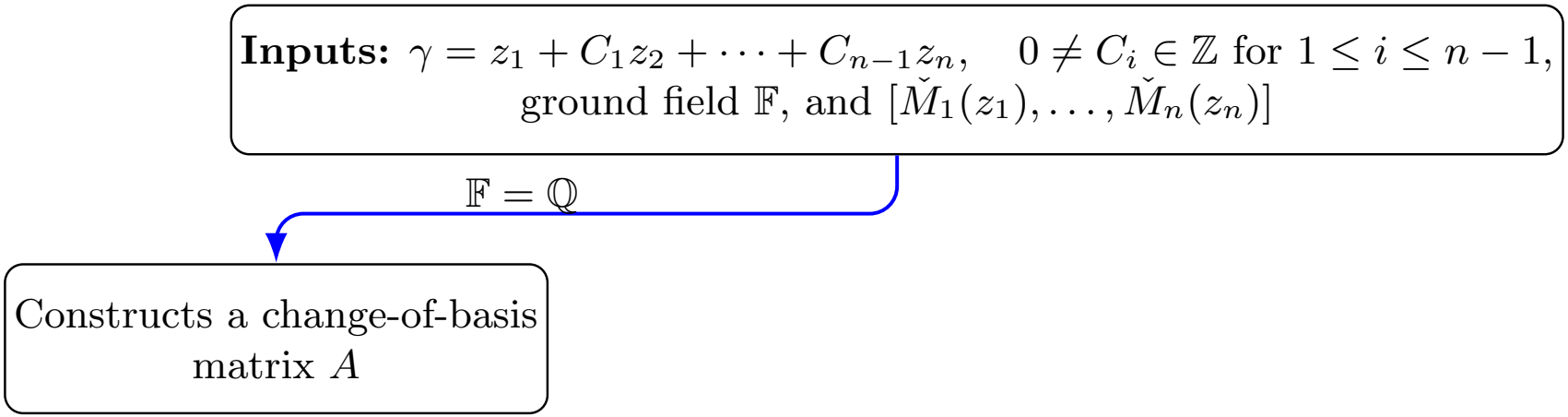
1. Let $\gamma = z_1 + C_1 z_2 + \ldots + C_{n-1} z_n$ be a candidate primitive element, where $0 \neq C_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$.
2. Let $B_\gamma = \{1, \gamma, \gamma^2, \ldots, \gamma^{d-1}\}$ be a potential basis for $\mathbb{Q}(\gamma)$.
3. Let B_L be a basis for $L \cong \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$.
4. We build the $d \times d$ change-of-basis matrix from B_γ to B_L

$$A = \left[\left[\begin{matrix} \vdots \\ 1 \\ \vdots \end{matrix} \right]_{B_L} \quad \ldots \quad \left[\begin{matrix} \vdots \\ \gamma^{d-1} \\ \vdots \end{matrix} \right]_{B_L} \right].$$

LAminpoly

Inputs: $\gamma = z_1 + C_1 z_2 + \cdots + C_{n-1} z_n, \quad 0 \neq C_i \in \mathbb{Z}$ for $1 \leq i \leq n-1$, ground field $\mathbb{F}$, and $[\check{M}_1(z_1), \ldots, \check{M}_n(z_n)]$

LAminpoly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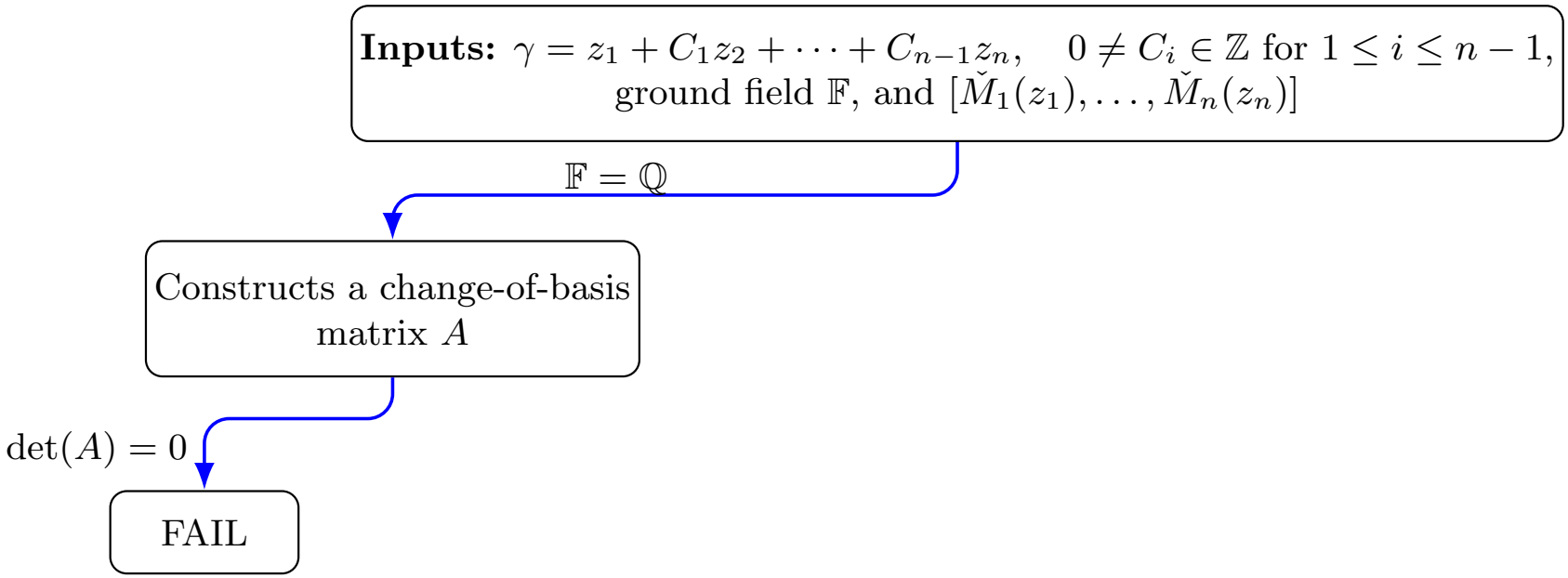
$\mathbb{F} = \mathbb{Q}$
Constructs a change-of-basis matrix A

LAminpoly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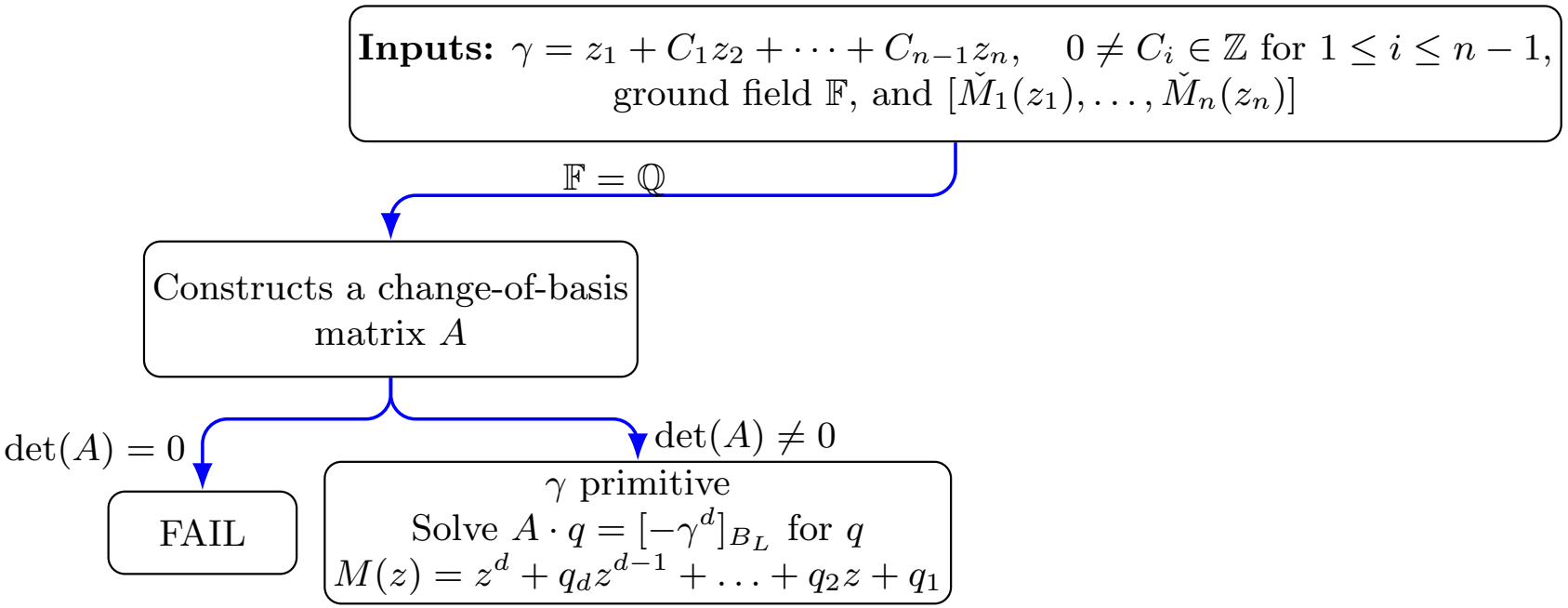
$\det(A) = 0$
FAIL

LAminpoly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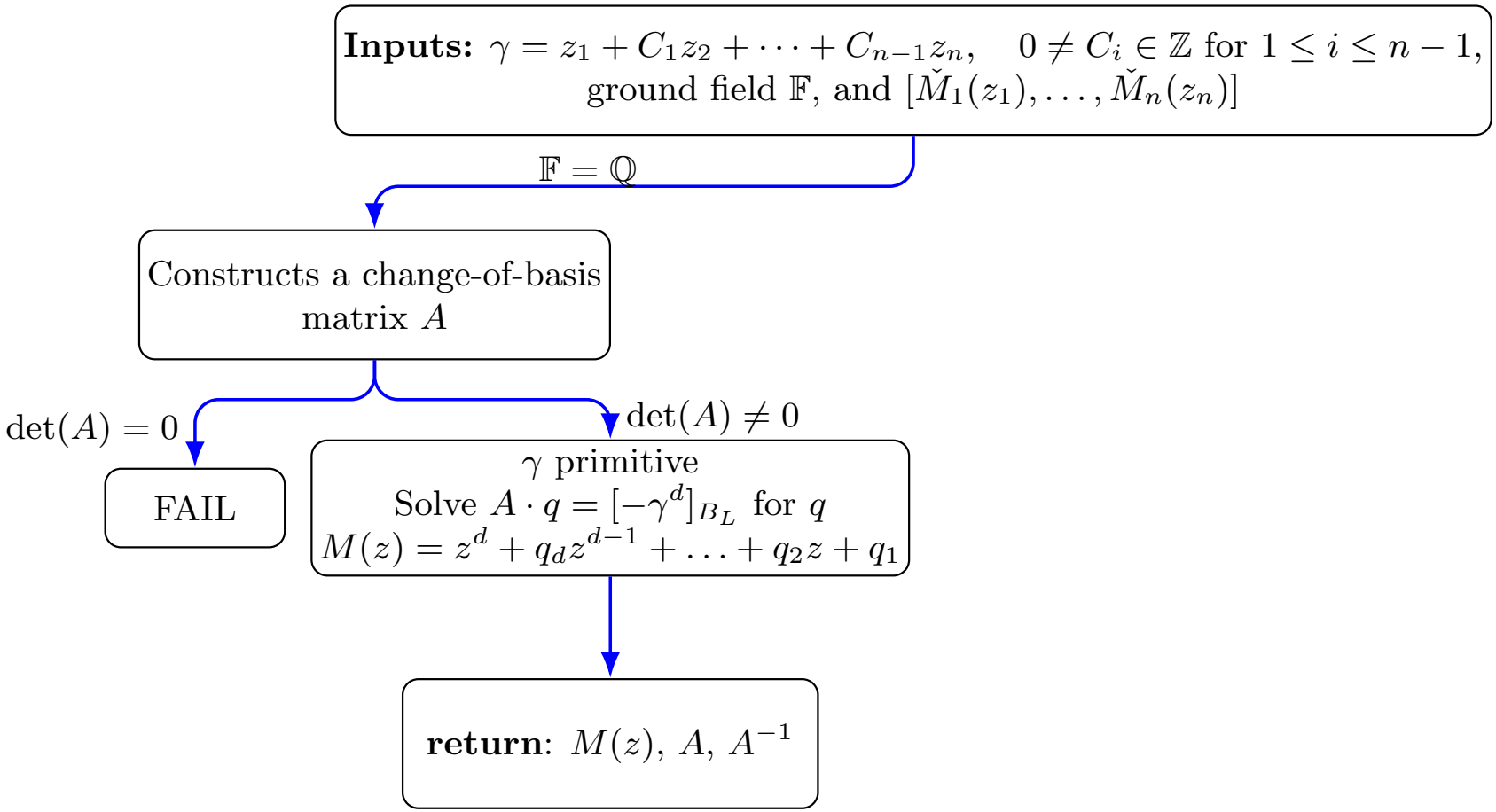
γ primitive
Solve $A \cdot q = [-\gamma^d]_{B_L}$ for q
$M(z) = z^d + q_d z^{d-1} + \ldots + q_2 z + q_1$

LAminpoly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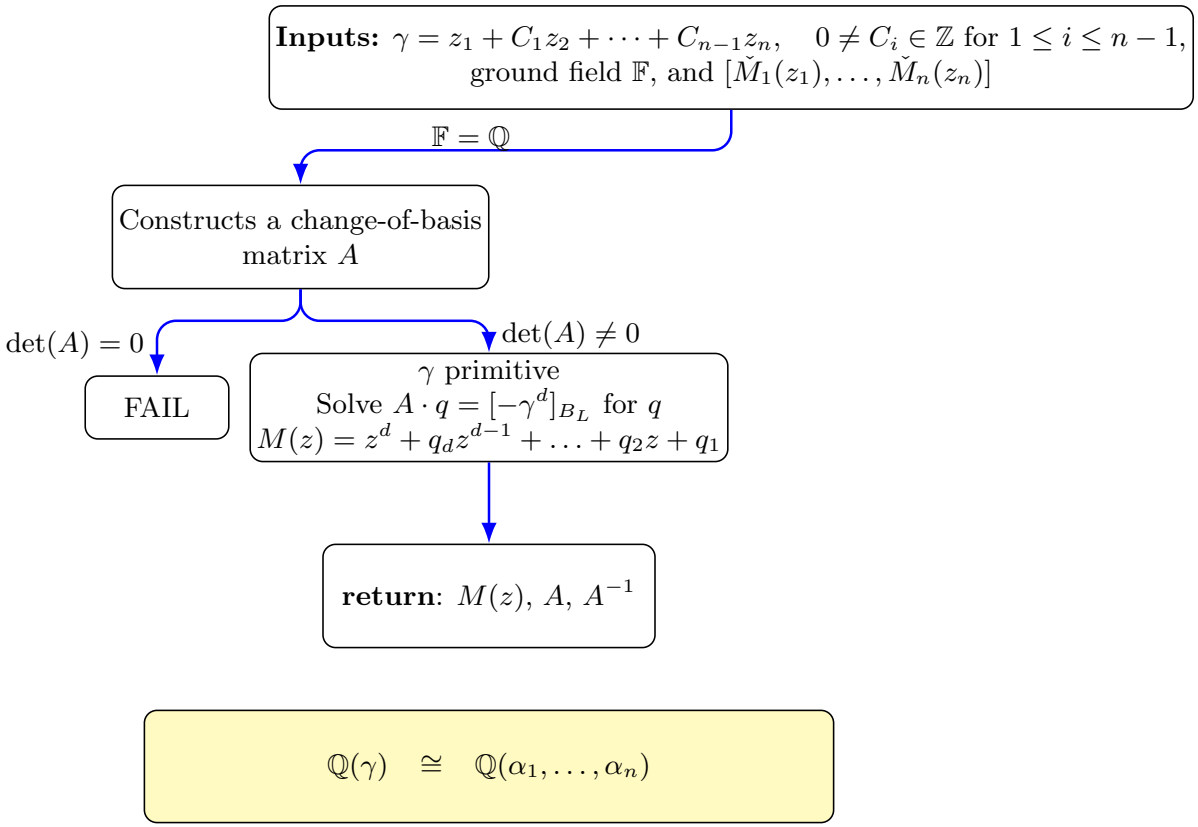
return: $M(z)$, A, A^{-1}

LAminpoly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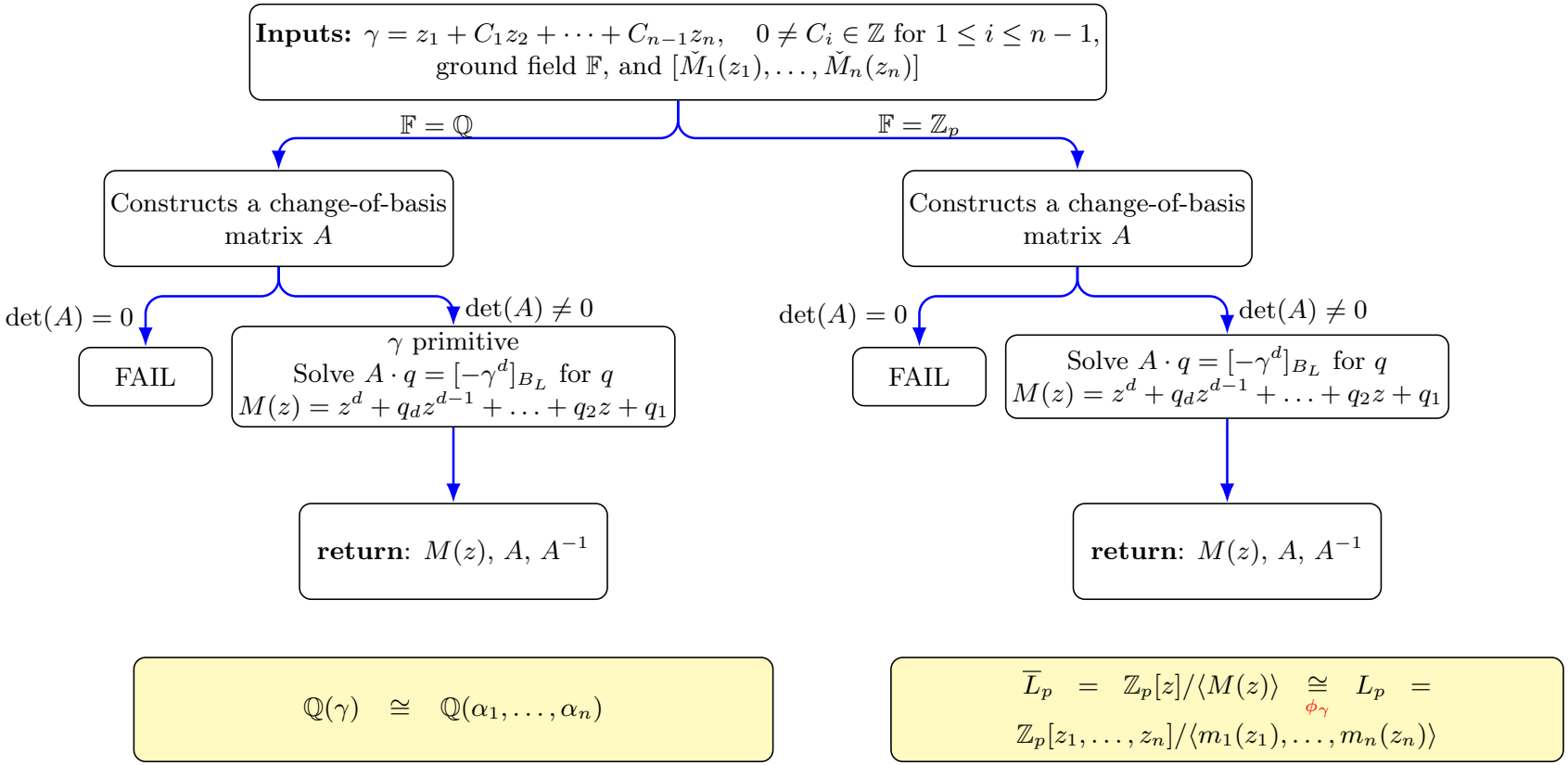
$M(z) = z^d + q_d z^{d-1} + \ldots + q_2 z + q_1$
return: $M(z)$, A, A^{-1}
$\mathbb{Q}(\gamma) \quad \cong \quad \mathbb{Q}(\alpha_1, \ldots, \alpha_n)$

LAminpoly



Remark

Running LAminpoly over $\mathbb{F} = \mathbb{Z}_p$:

- ✓ We avoid expression swell over $\mathbb{Q}$.
- ✗ It is likely that $m_i(z_i)$ is reducible over $\mathbb{Z}_p$ for some $1 \leq i \leq n$.

Example over $\mathbb{F} = \mathbb{Z}_p$

Let $L_7 = \mathbb{Z}_7[z_1, z_2]/\langle z_1^2 - 2, z_2^2 - 3\rangle$, where

$$z_1^2 - 2 \mod 7 = (z_1 - 3)(z_1 + 3) \in \mathbb{Z}_7[z_1].$$

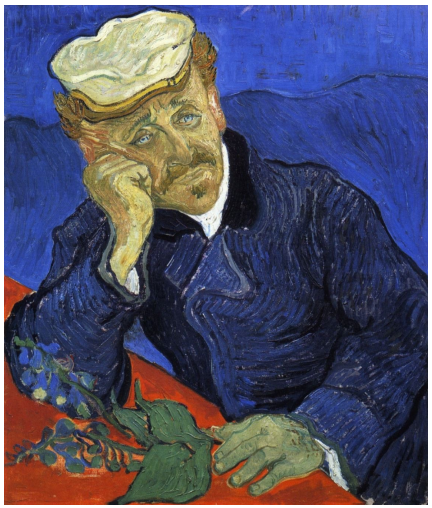
Let $B_L = \{1, z_2, z_1, z_1 z_2\}$ be a basis for L_7, $\gamma = z_1 + 2z_2$, and $B_\gamma = \{1, \gamma, \gamma^2, \gamma^3\}$. Then $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 4 & 0 \end{bmatrix}$, with $\det(A) \mod 7 = 1 \neq 0$ so $A \cdot q = (-[\gamma^4]_{B_L}) \Rightarrow q = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ so we build $M(z) = z^4 + 2 = (z^2 + 6z + 4)(z^2 + z + 4) \in \mathbb{Z}_7[z]$ and the ring $\overline{L}_7 = \mathbb{Z}_7[z]/\langle z^4 + 2\rangle$ s.t.

$$\overline{L}_7 \underset{\phi_\gamma}{\cong} L_7.$$

Problem

Let $f_1, f_2 \in \mathbb{Q}(\alpha_1, \ldots, \alpha_n)[x_1, \ldots, x_k]$, where $n > 1$ and $k > 1$.

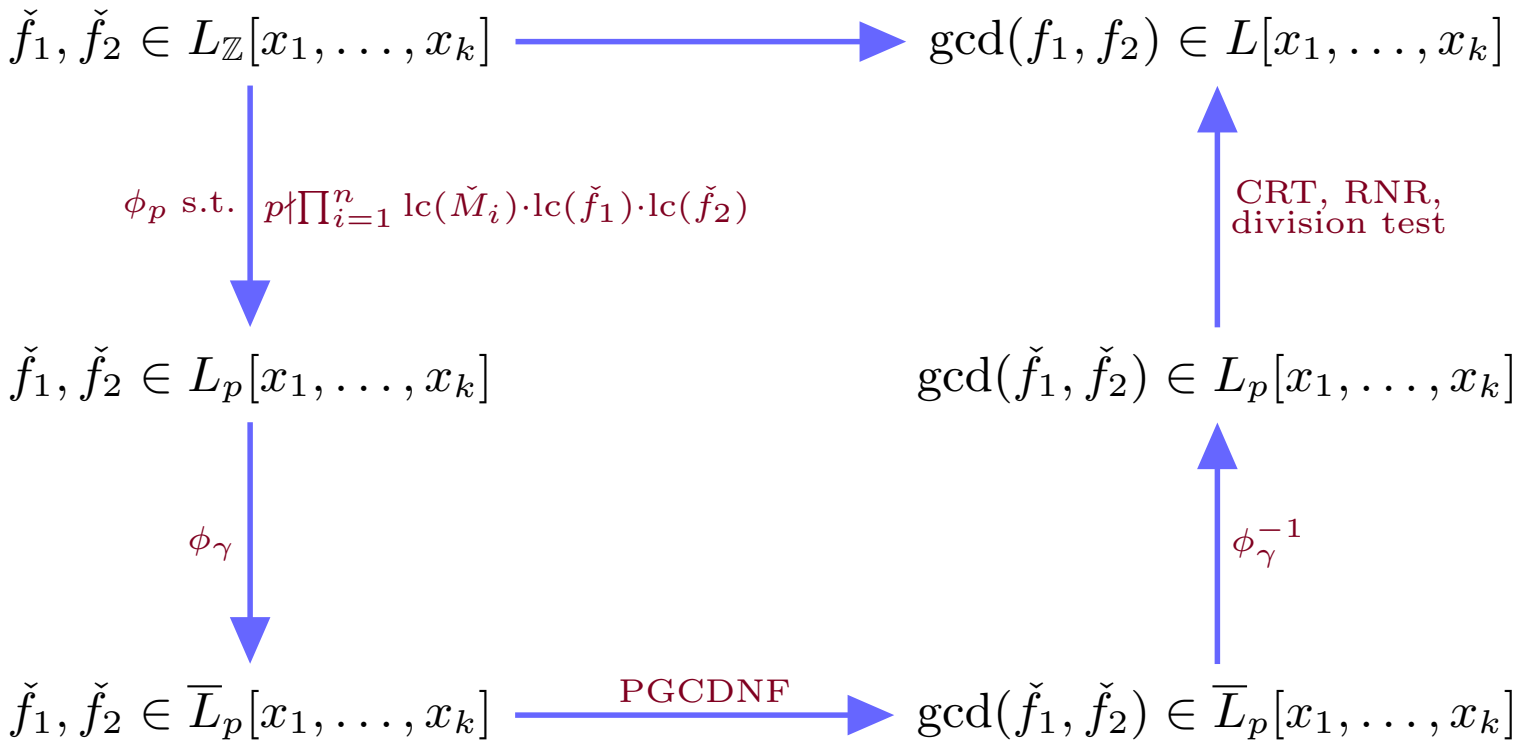
Problem: How to apply ϕ_γ to compute the monic $\gcd(f_1, f_2)$?



Outline

MGCDNF

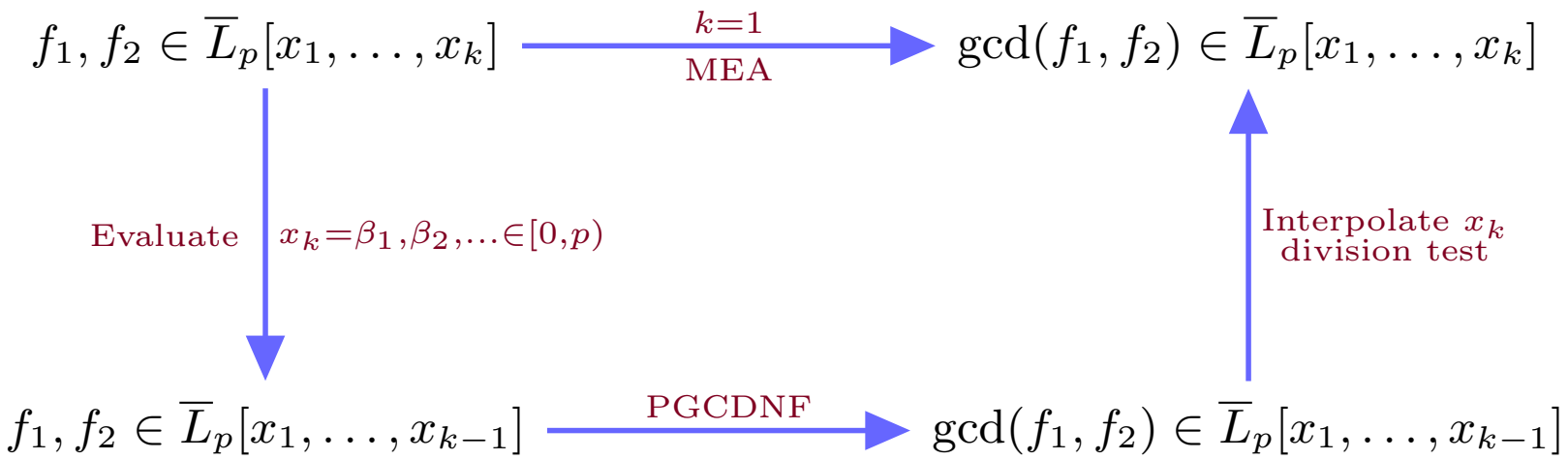
Let $f_1, f_2 \in L[x_1, \ldots, x_k]$, MGCDNF computes $g = monic \gcd(f_1, f_2)$.



$L = \mathbb{Q}[z_1, \ldots, z_n]/\langle M_1(z_1), \ldots, M_n(z_n)\rangle$
$L_{\mathbb{Z}} = \mathbb{Z}[z_1, \ldots, z_n]$
$L_p = \mathbb{Z}_p[z_1, \ldots, z_n]/\langle m_1(z_1), \ldots, m_n(z_n)\rangle$ s.t $m_i(z_i) = \check{M}_i(z_i) \mod p$
$\overline{L}_p = \mathbb{Z}_p[z]/\langle M(z)\rangle$

PGCDNF

Algorithm PGCDNF is a recursive algorithm to find the monic gcd of two polynomials f_1, f_2 in $\overline{L}_p[x_1, \ldots, x_k]$ where p is prime and $k \geq 2$.



Remark

MEA fails over $\overline{L}_p = \mathbb{Z}_p[z]/\langle M(z) \rangle$ if a zero divisor is encountered.

Lc-bad, Det-bad, Zero-divisor, and Unlucky

Question: Does every choice of prime or evaluation point lead to a successful reconstruction of the GCD?

Lc-bad, Det-bad, Zero-divisor, and Unlucky

Question: Does every choice of prime or evaluation point lead to a successful reconstruction of the GCD?
Answer: No!!!

- Lc-bad pairs (Are ruled out in advance)
- Det-bad pairs (Restart)
- Zero divisor pairs (Restart)
- Unlucky primes (Restart)
- Unlucky evaluation points (Restart)

Det-bad pairs

Definition(Det-bad pair)

Let p be a prime s.t. $p \nmid \prod_{i=1}^{n} \mathrm{lc}(\check{M}_i, z_i)$, $C = (C_1, \ldots, C_{n-1}) \in [1, p)^{n-1}$, and $\gamma = z_1 + \sum_{i=2}^{n} C_{i-1} z_i$. Let A be the change-of-basis matrix obtained from the LAminpoly over $\mathbb{F} = \mathbb{Q}$. The ordered pair (p, C) is called a **det-bad** pair if $\det(A) = 0$ or $p \mid \mathrm{numerator}(\det(A))$.

Example

Let $L = \mathbb{Q}[z_1, z_2]/\langle z_1^2 - 2, z_2^2 - 3\rangle$ and $B_L = \{1, z_1, z_2, z_1 z_2\}$. Let $C = (C_1) = (2)$, $\gamma = z_1 + C_1 z_2 = z_1 + 2z_2$, and $B_\gamma = \{1, \gamma, \gamma^2, \gamma^3\}$. Then the change-of-basis-matrix from B_γ to B_L is

$$A = \begin{bmatrix} 1 & 0 & 14 & 0 \\ 0 & 1 & 0 & 38 \\ 0 & 2 & 0 & 36 \\ 0 & 0 & 4 & 0 \end{bmatrix}, \text{ with } \det(A) = 160 = 2^5 \times 5, \text{ so the pairs}$$

$(p, C) = (2, (2))$ and $(p, C) = (5, (2))$ are det-bad pairs.

Benchmark 1: Degree of the field and number of extensions

Table: **Timings in CPU seconds for computing $\gcd(f_1, f_2)$ over an algebraic number field of degree d with n extensions, using recden.**

d	n	N	NMEA	D_f	D_g	MGCDNF 1			MGCDNF 2	
						time	LAMP	PGCDNF	time	PGCDNF
4	2	4	128	32	16	2.422	0	2.313	10.547	10.516
8	3	4	132	32	16	3.047	0	2.875	48.453	48.297
16	4	4	136	32	16	3.703	0.126	3.408	133.391	133.313
24	3	4	136	32	16	5.360	0.265	4.749	272.093	271.999
32	5	4	136	32	16	5.375	0.829	4.346	521.438	521.376
64	6	4	136	32	16	10.000	3.859	5.797	> 1000	(-)
128	7	4	136	32	16	32.422	20.951	10.781	> 1000	(-)
256	8	5	170	32	16	154.454	126.547	26.468	> 1000	(-)
512	9	4	136	32	16	542.906	491.140	48.890	> 1000	(-)
1024	10	(-)	(-)	32	(-)	> 1000	(-)	(-)	> 1000	(-)

Benchmark 2: Degree of the GCD (MGCDNF)

Table: **Timings in CPU seconds for computing $\gcd(f_1, f_2)$ over an algebraic number field of degree $d = 64$ with $n = 6$ extensions, using recden.**

d	n	N	$NMEA$	D_f	D_g	MGCDNF 1			MGCDNF 2	
						time	LAMP	PGCDNF	time	PGCDNF
64	6	3	18	4	2	3.265	3.032	0.123	7.203	7.188
64	6	4	40	8	4	4.984	3.874	0.796	51.984	51.937
64	6	4	56	12	6	5.250	3.764	1.188	62.328	62.281
64	6	4	72	16	8	5.469	3.984	1.283	274.359	274.297
64	6	4	88	20	10	6.766	3.922	2.502	321.250	321.173
64	6	4	104	24	12	7.359	4.108	2.938	542.750	542.579
64	6	4	116	28	14	9.609	4.249	4.876	> 1000	(-)
64	6	5	170	32	16	14.172	4.908	8.860	> 1000	(-)

Complexity of MGCDNF

Let $f_1, f_2 \in L[x_1, \cdots, x_k]$ and g be the monic $\gcd(f_1, f_2)$. The quantities involved in the running time of the MGCDNF algorithm are as follows:

- N is the number of good primes to reconstruct the monic g,
- $d = [L : \mathbb{Q}]$ and $D = \max_{i=1}^{k}(\deg(f_1, x_i), \deg(f_2, x_i))$,
- $H_M = \log_2 \max_{i=1}^{n} H(\check{M}_i)$, and $C = \log_2 \max(H(\check{f}_1), H(\check{f}_2))$.

Theorem

The expected time complexity of the MGCDNF algorithm is

$$\mathcal{O}\Big(\underbrace{Nd(nH_M + CD^k)}_{\text{reducing mod } p} + \underbrace{(Nd^3 + Nd^2D^k)}_{\phi_\gamma} + \underbrace{Nkd^2D^{k+1}}_{PGCDNF} + \underbrace{N^3dD^k}_{CRT/RNR} + \underbrace{(d^2D^2 + dD^k + dCD^k)}_{\text{Division test}}\Big).$$

1. M. Ansari and M. Monagan,
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Thank you!