# Equiangular Lines in Real and Complex Spaces


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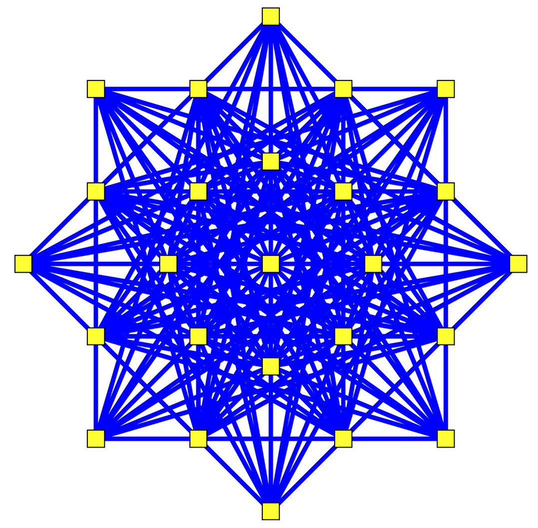


## The Theory

One of the most challenging problems in combinatorics is finding a large set of lines with few angles. In particular, the problem of finding equiangular lines in real and complex spaces is still wide open.

- We can specify each line by a unit vector $u$. Note that $-u$ represents the same line.
- The angle between unit vectors $u, v$ in $\mathbb{R}^d$ or $\mathbb{C}^d$ is determined by $|u^*v|$, the absolute value of their inner product.

**A set of lines in $\mathbb{R}^d$ or $\mathbb{C}^d$ spanned by unit vectors $x_1, \ldots, x_k$ is *equiangular* if there exists a constant $c$ such that $|x_i{}^* x_j| = c$ for every $1 \le i < j \le k$.**

From now on, let $X = \{x_1, \ldots, x_k\}$ be a set of equiangular lines.

### Equiangular lines in REAL SPACES ($X \subset \mathbb{R}^d$):

**Absolute bound:** $|X| \le \binom{d+1}{2}$

*Proof.* The symmetric matrices $x_1 x_1{}^T, \ldots, x_k x_k{}^T$ are linearly independent. The vector space of symmetric matrices has dimension $\binom{d+1}{2}$. □

It is known that if $|X| > 2d$ then the common angle must be of the form $\arccos \frac{1}{2k+1}$, $k \in \mathbb{Z}$. Also, if $|X| = \binom{d+1}{2}$ and $d > 3$ then $d + 2$ must be the square of an odd integer.

The maximum number of equiangular lines in $\mathbb{R}^d$, say $m(d)$, is only known for $d \le 43$, except for $d = 19, 20$. The value of this function is studied for over 45 years now, but less is known about it.

| $d$ | 2 | 3 | 4 | 5 | 6 | 7 | $\cdots$ | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | $\cdots$ | 42 | 43 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $m(d)$ | 3 | 6 | 6 | 10 | 16 | 28 | $\cdots$ | 28 | 36 | 40 | 48 | 48 | 72-76 | 90-96 | 126 | 176 | 276 | $\cdots$ | 276 | 344 |

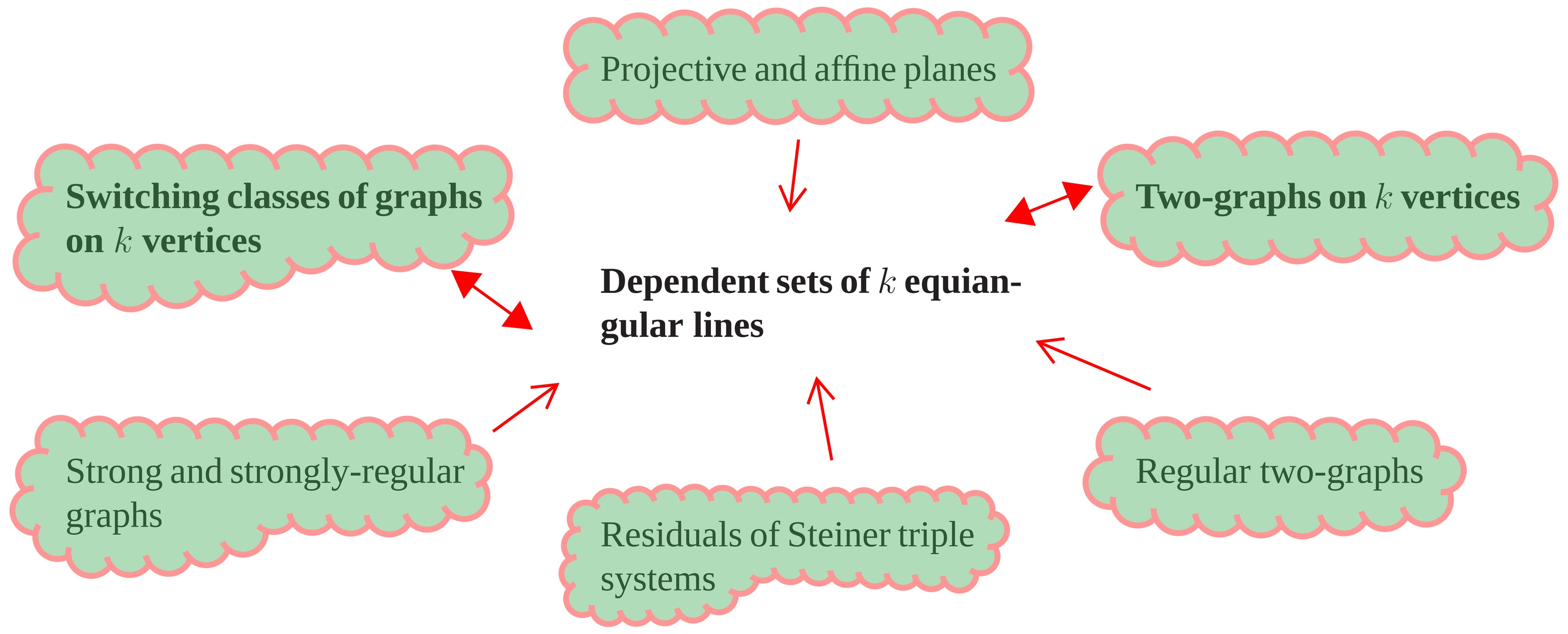
**Best known lower bound:** $m(d) \ge d^{3/2}$.

**The only known quadratic lower bound:** Dom de Caen [1] has a class of examples (using a specific family of codes) that provides a set of $2q^2$ equiangular lines in $\mathbb{R}^{3q-1}$, where $q = 2^{2t-1}$.

### Equivalent combinatorial objects:

A set of $k$ equiangular lines in $\mathbb{R}^d$ with common angle $\arccos \frac{1}{2k+1}$ ⟷ A graph on $k$ vertices where smallest Seidel eigenvalue $-(2k+1)$ has multiplicity $k - d$.

### Construction of real equiangular lines:



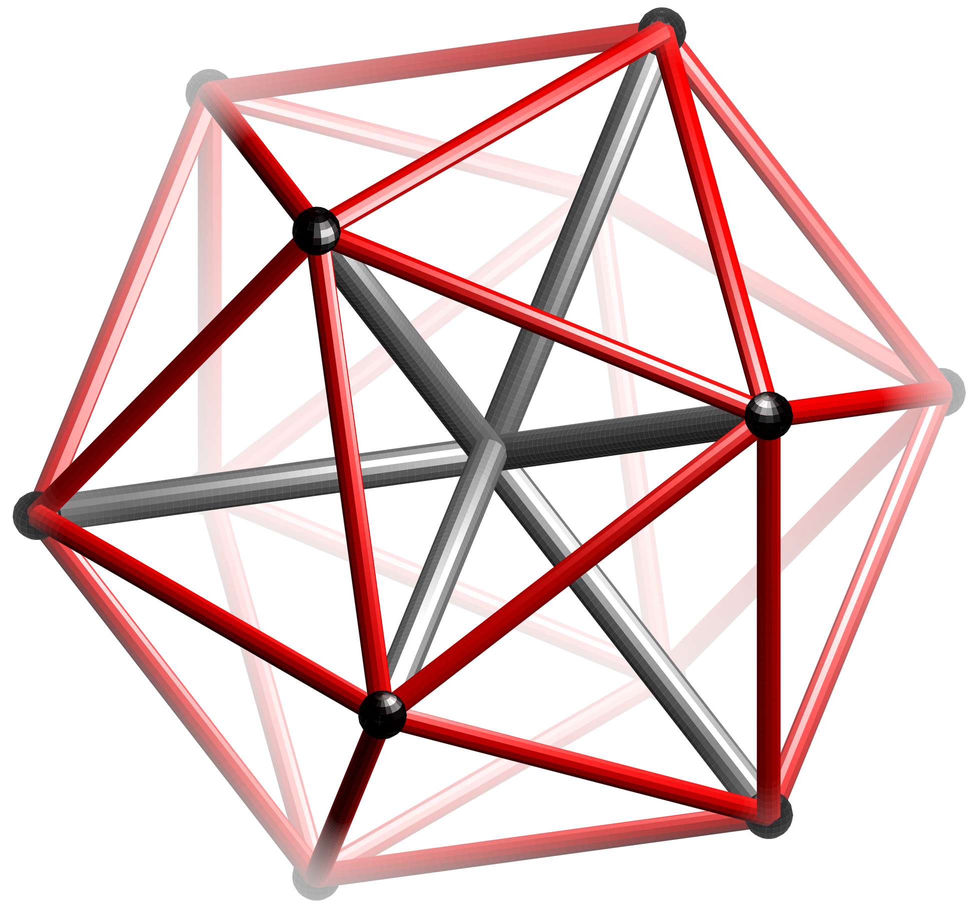



Figure 1: Six diagonals of Icosahedron form a set of equiangular lines in $\mathbb{R}^3$

### Equiangular lines in COMPLEX SPACES ($X \subset \mathbb{C}^d$):

**Absolute bound:** $|X| \le d^2$

*Proof.* The hermitian matrices $x_1 x_1{}^*, \ldots, x_k x_k{}^*$ are linearly independent. The vector space of hermitian matrices has dimension $d^2$. □

It is known that if $|X| = d^2$ then the common angle must be $c = \frac{1}{\sqrt{d+1}}$.

**Equivalent name in Physics:** Symmetric Informationally Complete Positive Operator-valued Measure (SIC-POVM). Their motivation to study these objects comes from quantum theory.

**OPEN PROBLEM:** Is it true that for all positive integers $d$, there is a set of $d^2$ equiangular lines in $\mathbb{C}^d$?

It is conjectured that the answer is *YES* and in fact a stronger result is believed to be true:

**Conjecture 1.** *For all $d$, there exists a vector $v \in \mathbb{C}^d$ and a group of Pauli matrices $G$ with $|G| = d^2$ such that the orbit $\{gv : g \in G\}$ forms a set of equiangular lines in $\mathbb{C}^d$.*

The above conjecture is known [2] to be true for $d \in \{2, 3, 4, 5, 6, 7, 8, 19\}$. Such a vector $v$ is called a *fiducial* vector.

**Theorem 1** (MK'2006). *Let $v = (r_j e^{i\theta_j})_{j \in \mathbb{Z}_d}$ be a fiducial vector in $\mathbb{C}^d$. Then the following identities hold:*

$$\sum_{j \in \mathbb{Z}_d} r_j^2 r_{j+s}^2 = \begin{cases} \frac{1}{d+1} & s \in \mathbb{Z}_d \setminus \{0\} \\ \frac{2}{d+1} & s = 0 \end{cases}$$

A set of vectors in $\mathbb{R}^d$ is called equiangular if the inner product between every two vector in the set is a constant. Note the difference of equiangular vectors with equiangular lines where you require the absolute value of the inner products to be a constant.

**Theorem 2** (MK'2006). *Assume that $v = (r_j e^{i\theta_j})_{j \in \mathbb{Z}_d}$ is a fiducial vector in $\mathbb{C}^d$. Let $x = (\sqrt{\frac{d+1}{2} r_j^2})_{j \in \mathbb{Z}_d}$, and $\sigma$ be the right shift operator. Then $S = \{x, \sigma x, \ldots, \sigma^{d-1} x\}$ is a set of $d$ equiangular vectors on the unit sphere in $\mathbb{R}^d$ with common angle $60°$. Moreover, the set $S$ has to be unique up to a unitary transformation.*

## Computation

**Searching for numerical fiducial vectors**

- One may observe that a fiducial vector is the global minimum of a certain homogenous polynomial: $\sum_{g \in G} |x^* g x|^4$.
- We have used the `Minimize` command in MAPLE to find such a vector with high precision. One of the challenges was to deal with many local minimum vectors that exist around the global minimum. The good news is that the (preconditioned) conjugate gradient method is implemented in the **Optimization** package in MAPLE and that arbitrary precision is supported.
- Using the above idea, P. Lisonek, A. Roy, and myself were able to prove that there are exactly six non-equivalent (up to phase shift, conjugation, rotation, and reflection) fiducial vectors in $\mathbb{C}^8$.
- Using the `PSLQ` algorithm, which is built-in MAPLE, we found the minimal polynomial of each coordinate of these fiducial vectors.

**Observation.** Let $z = re^{i\theta}$ be one of the coordinates of a fiducial vector in $\mathbb{C}^d$. We observed that the degree of the minimal polynomial of $z$ grows exponentially in $d$. However, the minimal polynomials of $r$ and $\tan\theta$ grow linearly in $d$. Specifically, the minimal polynomial of $r$ has only even terms, thus one may look at the minimal polynomial of $r^2$. We also observed that certain scalars of $r^2$ (depending on $d$) have minimal polynomials with significantly smaller coefficients. Analogously, the minimal polynomial of $\tan\theta$ has also even terms and is 'almost' reciprocal.

**Example.** Here is an example of a minimal polynomial of $12r^2$, where $r$ is the absolute value of the first coordinate of one of the six fiducial vectors in $\mathbb{C}^8$:

$$\begin{aligned} & x^{16} - 32\,x^{15} + 468\,x^{14} - 4128\,x^{13} + 24462\,x^{12} - 103088\,x^{11} \\ & + 320288\,x^{10} - 751328\,x^{9} + 1343683\,x^{8} - 1813296\,x^{7} + 1789664\,x^{6} \\ & - 1240160\,x^{5} + 597150\,x^{4} - 229520\,x^{3} + 91580\,x^{2} - 29792\,x + 4049 \end{aligned}$$

Here is the same polynomial for the second coordinate:

$$x^2 - 3x + 1$$

In fact, the minimal polynomial of each of the other coordinates is one of the above!

Here is the minimal polynomial of $\tan^2\theta$, where $\theta$ is the argument of one of the coordinates of one of the six fiducial vectors in $\mathbb{C}^8$:

$$x^8 - 408x^7 + 7752x^6 - 48168x^5 + 124578x^4 - 144504x^3 + 69768x^2 - 11016x + 81$$

You may observe that the coefficient of $x^i, (i = 0, \ldots, 4)$ is equal to the coefficient of $x^{8-i}$ times $3^{4-i}$.

## References


[1] D. DE CAEN, *Large equiangular sets of lines in Euclidian space*, J. Combin., 7 (2000), Research Paper 55, 3 pp. (electronic).

[2] J. M. RENES, R. BLUME-KOHOUT, A. J. SCOTT, AND C. M. CAVES, *Symmetric informationally complete quantum measurements*, J. Math. Phys., 45 (2004), pp. 2171–2180.


*The graph shown at the top right is the complement of line graph of the complete graph on $8$ vertices. The smallest Seidel eigenvalue of this graph is $-3$ with multiplicity $21$. This proves the existence of $28$ equiangular lines in $\mathbb{R}^7$.