

$$\int p_1 e^{\theta^{-1}} + \dots + p_0 + \dots + p_m \theta^m = q_1 e^{\theta^{-1}} + \dots + q_0 + \dots + q_m \theta^m + \Sigma L$$

Equating coefficients in θ^j yields

$$j \neq 0 \quad p_j = q_j' + j w q_j \quad \Leftrightarrow \quad p_j \theta^j = [q_j \theta^j]'$$

$$j=0 \quad p_0 = q_0' + \Sigma L'$$

The case $j=0$ $\int p_0$ where $p_0 \in F$ is an \int problem in F which can be solved recursively. Otherwise we must solve

$$\overset{F}{\underset{w}{q}}_j' + j \overset{F}{\underset{w}{w}} q_j = \overset{F}{\underset{w}{p}}_j \quad \text{for } q_j \in F = K(x)(\theta_1, \dots, \theta_n).$$

This is called a Risch differential equation.

If it has no solution in F then $\int p_j$ is not elementary.

Example $\int x e^{x^2} dx$ $F(\theta) = \mathbb{Q}(x)(e^{x^2})$ $w = x^2$
 $w' = 2x$

$$\int x \theta dx = \overset{F}{\underset{w}{q}}_1 \theta + \text{constant}$$

$$\Rightarrow x \theta = q_1' \theta + q_1 2x \theta = (q_1' + 2x q_1) \theta$$

$$\Rightarrow x = q_1' + 2x q_1 \quad \text{and } q_1 \in \mathbb{Q}(x)$$

$$\Rightarrow q_1 = \frac{1}{2} \quad \text{by inspection}$$

$$\int x e^{x^2} dx = q_1 \theta = \frac{1}{2} e^{x^2}$$